Consumption and Wealth Tails in Heterogeneous-Agent Models: The Pareto Trap

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Abstract

Household wealth is more concentrated at the top than household consumption. In weighted surveys for Australia, Germany, Italy, and the Netherlands, the fitted consumption tail index exceeds the wealth index, and three of four index gaps survive a prespecified multiple-comparison adjustment. Benchmark heterogeneous-agent models cannot reproduce this ranking: whenever optimal consumption remains above a fixed positive fraction of wealth, a Pareto-type wealth tail keeps the consumption tail at least as heavy, a *Pareto trap*. The restriction extends to a finitely lived overlapping-generations economy. Calibrated to each eligible wealth index, the benchmark's untargeted consumption predictions miss all four point estimates.

Keywords: Heterogeneous-agent models, Pareto tail index, consumption, wealth
JEL Classification: D31, E21, C63

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1 Introduction

Household wealth is substantially more concentrated at the top than household consumption. Write α_C and α_W for the population Pareto tail indices and $\hat{\alpha}_C$ and $\hat{\alpha}_W$ for their estimates. A smaller index means a heavier tail. Existing U.S. evidence places the wealth index near 1.5 and the consumption index around 3 (Klass et al., 2007; Vermeulen, 2016; Gaillard et al., 2023). The estimated ranking is therefore $\hat{\alpha}_C > \hat{\alpha}_W$. This paper asks what that ranking implies for heterogeneous-agent models calibrated to the wealth tail.

The key observation is an overidentifying restriction. Suppose a model keeps consumption above a positive fraction of a household resource variable at high resource levels. Then households in the resource tail necessarily populate the consumption tail, so consumption cannot have a strictly thinner Pareto tail. Once the model is disciplined to match α_W, it cannot choose an arbitrary α_C while this exposure persists. Matching the wealth tail is therefore not sufficient model validation. It creates a second, untargeted implication for consumption.

The paper makes three contributions. First, it estimates both tail indices in three household survey systems spanning eight countries (HILDA, the euro-area HFCS, and CHIP), complementing the U.S. evidence in Gaillard et al. (2023). The fitted consumption index exceeds the fitted wealth index in every weighted-survey comparison, four index pairs pass the prespecified eligibility checks, and the difference survives the prespecified Holm adjustment for Australia, Germany, and Italy (Section 2). The two Chinese nonhousing-resource comparisons are reported over the observed range only. These estimates and their eligibility rules are fixed before any model comparison.

Second, Theorem 3.1 derives the tail restriction from one property of the consumption policy. If $C \geqslant aX$ once resources X are sufficiently large, the event $\{X > c/a\}$ is contained in $\{C > c\}$. Thus a Pareto resource tail with index α_X rules out a Pareto consumption tail with $\alpha_C > \alpha_X$. The proof requires neither independence, homotheticity, nor asymptotic linearity, and it does not assume that consumption itself has a Pareto law. The pointwise policy bound can also be weakened to a nonvanishing exposure among households in the resource tail. The observable converse is equally useful. If $\alpha_C > \alpha_X$, then among households with $X > x$, the fraction consuming more than any fixed positive share of X must converge to zero (Corollary 3.4). The marginal tail comparison therefore has a conditional policy diagnostic.

Two complementary sharpenings clarify the scope. Feasibility supplies the reverse bound: under the maintained positive lower bound, if consumption also cannot exceed wealth and wealth has a Pareto-type tail, consumption inherits the index as a

weak Pareto-type tail, with arbitrary dependence between wealth and the exogenous state (Corollary 4.5). Independence plus an exact Pareto wealth tail and an affine or concave asymptotically linear policy deliver a Pareto tail for consumption with an explicit constant (Proposition 3.3). The two results should not be conflated: the first handles Markov dependence, while the second proves the stronger distributional conclusion.

Third, Section 4 shows why the restriction binds in a standard income-fluctuation model. Condition (19) of Ma et al. (2020) implies $C \geqslant (1 - \bar{x})W$, while their cash-on-hand choice set gives $C \leqslant W$. Together, these inequalities imply that consumption inherits the weak wealth-tail index whenever wealth has a Pareto-type tail, with arbitrary Markov dependence (Corollary 4.5). The same stability certificate is maintained by the model's stationarity and heavy-tail argument. The route used to support a stationary heavy wealth tail therefore prevents a thinner Pareto consumption tail at the same time. This circular dependence is the *Pareto trap*.

A second class of applications is covered by the affine sufficient condition. In the finitely lived OLG economy of Benhabib et al. (2011), optimal consumption is affine in wealth with state-dependent coefficients and a bounded human-wealth intercept, so the lower bound holds even though consumption can exceed financial wealth, and the model's own population power-tail bounds transfer to consumption with the same exponent (Proposition 4.6). Appendix B records, model by model, where the positive linear lower bound is a theorem and where it remains conjectural.

The quantitative magnitude can be large. For a full Pareto distribution with index $\alpha > 1$, the share held by the top p fraction is $p^{1-1/\alpha}$. Replacing $\alpha_C = 3.0$ by $\alpha_W = 1.5$ raises the implied top 1% consumption share from 4.6% to 21.5%, a factor of about 4.6. An empirical population share also depends on the contribution below the fitted threshold. Section 5 uses calibrated policies and an explicitly auxiliary tail calculation to illustrate the mechanism without treating a globally linearized wealth process as a finite-wealth model fit.

The closest parametric comparison is Gaillard et al. (2023). In a continuous-time economy, they derive complete mappings among consumption, labor-income, wealth, and capital-income tails from asymptotic policy formulas and an endogenous tail-factorization result. In deriving the wealth tail in Proposition 3, they verify that the stationary joint density factors at high wealth into a Pareto term and a productivity term (their footnote 27), and Theorem 3.1 here allows arbitrary dependence between wealth and the exogenous state. The present result instead gives a one-sided policy restriction and a discrete-time stability certificate. It does not claim to remove an independence assumption from, or subsume, their framework. The lower-bound argument also relates to the impossibility result of Stachurski and Toda (2019), under

the corrected assumptions in Stachurski and Toda (2020). They show that canonical risk-free Bewley–Huggett–Aiyagari models cannot generate wealth tails heavier than income. Here the same type of consumption lower bound is used in the other direction: conditional on a heavy resource tail, it restricts consumption. The paper thereby connects work on tail generation in heterogeneous-agent economies (Benhabib et al., 2011, 2015; Benhabib and Bisin, 2018; Nirei and Aoki, 2016; Jones and Kim, 2018; Gabaix et al., 2016; Beare and Toda, 2022; Achdou et al., 2022) to evidence on consumption and wealth inequality (Krueger and Perri, 2006; Aguiar and Bils, 2015; Attanasio and Pistaferri, 2016; Meyer and Sullivan, 2017).

The scope is deliberately impossibility-focused. The paper does not add a structural escape mechanism and does not classify multi-asset or HANK models without a model-specific lower-exposure certificate. Prominent two-asset and HANK benchmarks therefore remain outside the proved class rather than being classified by analogy (Kaplan and Violante, 2014; Kaplan et al., 2018). The proof-status tables in Appendix B separate formally established applications from conjectural and open cases. Section 2 presents the empirical tails, Section 3 proves the restriction and its sharpenings, and Section 4 verifies the applications. Section 5 gives the quantitative diagnostic, and Section 6 concludes. Data and estimation details appear in Appendix D.

2 Empirical tails of wealth and consumption

Before turning to the estimates, we formalize the tail notation used in the introduction.

Definition 2.1 (Pareto tail and tail index)**.** For a real-valued random variable X with distribution F_X, write $\bar{F}_X(x) := \mathbb{P}\{X > x\} = 1 - F_X(x)$ for the survival function, or complementary cumulative distribution function (CDF). The distribution F_X has a *Pareto tail* with index $\alpha > 0$ if there exists $k > 0$ such that

$$\lim_{x \to \infty} x^{\alpha} \bar{F}_X(x) = k. \tag{1}$$

A smaller index means a heavier tail. When the indices exist, α_C and α_W denote the consumption and wealth tail indices.

This section estimates both tail indices of the wealth and consumption distributions in three household survey systems and fixes in advance which estimates may discipline the models of Sections 4 and 5. The data are the Household, Income and Labour Dynamics in Australia survey (HILDA, wave 22), the euro-area Household Finance and Consumption Survey (HFCS, wave 5, six countries), and the China Household Income Project (CHIP, 2013 and 2018). Each survey records consumption and

wealth for the same households, so the two margins are estimated jointly rather than matched across sources. For the United States, Gaillard et al. (2023) document the same consumption and wealth ranking directly, and Toda and Walsh (2015) establish the power-law behavior of the consumption tail itself.

Consumption is each survey's annual household expenditure aggregate: an eighteen-component basket of nondurable goods and services in HILDA, expenditure on consumer goods and services in the HFCS, and total consumption expenditure in CHIP. The three concepts differ in breadth across the surveys rather than forming one nested definition, and the ordering reported below appears under each of them.

Wealth is household net worth in HILDA and the HFCS. The CHIP resource is nonhousing assets net of matched liabilities, a narrower concept treated as a separate replication rather than a comparable net worth. HFCS values are converted to 2024 prices using each household's fieldwork year, while HILDA waves and CHIP years are estimated separately in source-year currency, so no cross-year price adjustment arises.

Each index is a Pareto maximum likelihood estimate on strict exceedances above the fixed 90th percentile of that variable's positive observations, computed with provider survey weights in HILDA and the HFCS and with equal household weights in CHIP, and corrected for first-order finite-sample bias.

Released HILDA wealth is top coded at official wave-specific cutoffs that overlap the fitted region, so the Australian wealth fit treats top-coded households as right-censored events (Wang and Xiao, 2022), with a truncated-window fit (Aban et al., 2006) reported as the local description. In wave 22 the official cutoff is \$8,000,000 while the fitted tail begins at \$3,063,000, so the cutoff sits at 2.6 times the threshold, and the tail fit carries 729 exactly observed households plus 112 censored at the cutoff.

We also hold access to the HILDA Restricted Release, in which wealth is not top coded, and use it below to validate the correction directly. Uncertainty is survey specific: HILDA uses 45 replicate weights, the HFCS supplies five implicates, that is, five completed versions of the survey data, each carrying 1,000 replicate weights, with estimates and covariance matrices combined across implicates by Rubin's rules (Rubin, 1987), and CHIP uses a paired bootstrap that resamples households separately within the urban and rural sectors. Appendix D records construction and estimator details.

Fitted indices support three claims of increasing strength: a description of the fitted range, an extrapolation to the tail beyond it, and a target a model may be calibrated to. The second and third require more than a good fit, so the design separates them in advance. The seven weighted-survey comparisons, Australia and the six HFCS countries, form one prespecified family for multiple testing.

An index pair enters asymptotic comparisons only if it passes a four-part eligibility rule covering construction, fit, threshold stability, and inference, and ineligible pairs are retained in the family with a p-value of one, so screening cannot manufacture significance. The one-sided hypothesis that the consumption index exceeds the wealth index is tested with a Holm adjustment at the 5 percent familywise level (Holm, 1979). The adjustment controls the probability of even one false rejection across the seven comparisons at the 5 percent level. Appendix D.2 defines the four checks and the step-down rule. The two Chinese samples form a separate family, reported over the observed range and not tested. A stricter exact-tail benchmark, which licenses calculations that assume a Pareto distribution outright, additionally requires eligible Pareto fits on both margins with both indices above one.

Table 1: Tail estimates and reporting status

Country and sample	Wealth/resource fit	$\widehat{\alpha}_C$ (s.e.)	$\widehat{\alpha}_W$ (s.e.)	$\widehat{\alpha}_C - \widehat{\alpha}_W$ (s.e.)
Australia (HILDA), wave 22	censored continuation	3.168 (0.244)	2.119 (0.159)	1.050 (0.199)*
	local truncated (u,c)	3.168 (0.244)‡	2.027 (0.308)‡	1.141 (0.242)
Germany (HFCS), wave 5	uncensored working fit	4.085 (0.220)	1.603 (0.132)	2.482 (0.276)*
Spain (HFCS), wave 5	uncensored working fit	3.076 (0.166)†	1.659 (0.080)†	1.417 (0.188)
Finland (HFCS), wave 5	uncensored working fit	4.685 (0.126)†	1.856 (0.057)†	2.830 (0.139)
France (HFCS), wave 5	uncensored working fit	2.740 (0.109)†	1.726 (0.060)†	1.014 (0.121)
Italy (HFCS), wave 5	uncensored working fit	3.383 (0.214)	1.615 (0.127)	1.768 (0.223)*
Netherlands (HFCS), wave 5	uncensored working fit	2.906 (0.235)	2.096 (0.132)	0.810 (0.271)
China (CHIP), 2013	nonhousing resource	2.830 (0.093)‡	1.818 (0.069)‡	1.012 (0.108)
China (CHIP), 2018	nonhousing resource	2.808 (0.057)‡	1.613 (0.036)‡	1.195 (0.065)

* marks rejection of a non-positive index difference after the fixed-family Holm adjustment defined in the text. † marks an index pair not qualified under the four-part eligibility rule. ‡ marks a window-local fit with no asymptotic index claim. Markers qualify the index pair or row, not either marginal fit in isolation. The unmarked Dutch pair is eligible, but its adjusted one-sided test does not reject a non-positive difference. The Chinese samples form a separate family and are not tested.

Table 1 reports the estimates and their reporting status. Four index pairs are eligible: Australia, Germany, Italy, and the Netherlands. The fitted consumption index exceeds the fitted wealth index in all seven weighted-survey comparisons, and the difference survives the Holm adjustment for Australia (gap 1.050, standard error 0.199), Germany (2.482, 0.276), and Italy (1.768, 0.223). The Dutch pair is eligible, and its adjusted test does not reject (adjusted p-value 0.081). Spain, Finland, and France show positive fitted differences but fail eligibility, so their rows are descriptive. The Chinese comparisons are positive in both years (1.012, standard error 0.108, in 2013 and 1.195, 0.065, in 2018) over the observed range only. Only Germany and Italy pass the exact-tail benchmark.

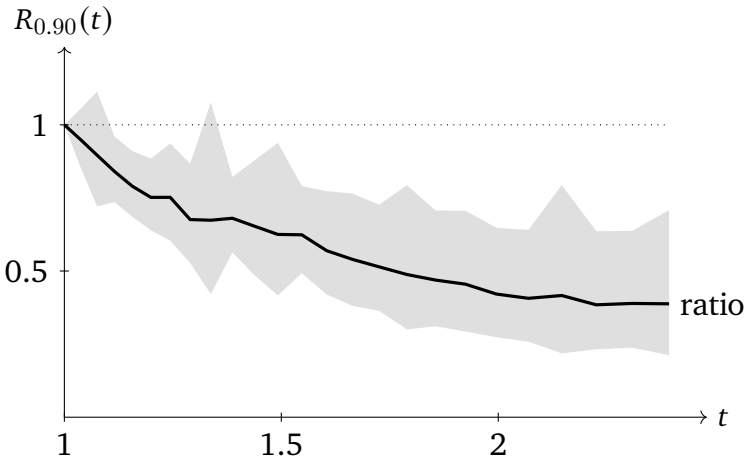

Figure 1: Survival-order profile for the Australian baseline (HILDA wave 22). The curve is the ratio of the two normalized survival functions above their own weighted 90th percentile anchors v_C and v_W, $R_{0.90}(t) = \dfrac{\mathbb{P}\{C \geq t\, v_C\}/\mathbb{P}\{C \geq v_C\}}{\mathbb{P}\{W \geq t\, v_W\}/\mathbb{P}\{W \geq v_W\}}$, plotted against the multiplier t. The shaded region is the released simultaneous 95 percent band for this profile. The anchors are taken on the full matched sample, so the wealth anchor, \$3,020,000, sits slightly below the \$3,063,000 positive-sample threshold of the index fit.

Figure 1 shows the Australian profile. The estimated ratio is below one at every plotted multiplier above the anchor. At the largest supported multiplier, 2.394, it is 0.388, with a simultaneous 95 percent band of 0.212 to 0.708.

Two further checks support the Australian correction and the German estimate. Rerunning the identical estimation procedure on the HILDA Restricted Release, where wealth is not top coded, reproduces the corrected wealth indices to within about 0.05 to 0.07 index points in each of the five matched waves, with the restricted estimate the slightly larger of the two every time, and no wave rejects equality of the pair after the prespecified two-sided Holm adjustment across the five waves. The matched results support the correction, and failure to reject is not proof of equivalence. Independently, König et al. (2025) obtain a German wealth index of 1.66 from the 2019 German Socio-Economic Panel including its high-wealth household sample, close to the 1.603 estimated here.

The four eligible pairs are the admissible model targets of this paper. The calibration discipline is fixed here: each eligible wealth index supplies a separate calibration target, and the model's consumption index is an untargeted prediction for comparison with the corresponding survey estimate. The evidence is a contemporary baseline: coverage spans eight countries, four samples pass the prespecified checks, and the earlier HFCS wave shows a different eligibility pattern, so the claim is about the current waves, not about every country in every wave.

3 The impossibility theorem

This section states the paper's central restriction. If stationary wealth has a Pareto-type tail and consumption is bounded below by a positive fraction of wealth at high wealth (Assumption 3.1), then stationary consumption cannot have a Pareto-type tail index strictly larger than that of wealth (Theorem 3.1). The economic work is therefore to identify conditions under which optimal policies satisfy this bound.

The simplest case illustrates the logic: if optimal consumption is a fixed fraction of wealth, independent of the exogenous state, the two distributions share the same tail index (Lemma A.1 with $\rho = 1$). More generally, an exact affine policy or a concave asymptotically linear policy, each conditional on the exogenous state, implies Assumption 3.1 (Proposition 3.2). If wealth has a Pareto tail, not merely a Pareto-type tail, and is independent of the exogenous state, then consumption has a Pareto tail with the same index and an explicit constant (Proposition 3.3).

3.1 Tail classes

Definition 2.1 fixed the Pareto tail through the limit $x^{\alpha}\bar{F}_X(x) \to k$. The theorem needs only the weaker requirement that the survival function behaves like a power under rescaling, and its corollaries use a weaker logarithmic index still. This subsection fixes both classes (Bingham et al., 1987).

Definition 3.1 (Regular variation)**.** A measurable function $J : (0,\infty) \to (0,\infty)$ is *regularly varying* with index $\eta \in \mathbb{R}$, written $J \in RV_\eta$, if

$$\frac{J(tx)}{J(x)} \to t^{\eta} \qquad \text{as } x \to \infty \quad \text{for every fixed } t > 0.$$

When $\eta = 0$, J is said to be *slowly varying*.

Definition 3.2 (Pareto-type and weak Pareto-type tails)**.** Let $X \geqslant 0$ and let $\alpha > 0$.

(i) X has a *Pareto-type tail* with index α if there exist $x_0 > 0$ and a slowly varying function L such that $\bar{F}_X(x) = x^{-\alpha}L(x)$ for every $x \geqslant x_0$. Equivalently, $\bar{F}_X \in RV_{-\alpha}$.

(ii) X has a *weak Pareto-type tail* with index α if there exists $x_0 > 0$ such that $\bar{F}_X(x) > 0$ for all $x \geqslant x_0$, and $-\log \bar{F}_X(x)/\log x \to \alpha$.

(iii) X has a *Pareto distribution* with index α and threshold $x_{\min} > 0$ if $\bar{F}_X(x) = (x_{\min}/x)^{\alpha}$ for all $x \geqslant x_{\min}$.

In each case, α is the tail index.

The hierarchy is: Pareto distribution implies Pareto tail implies Pareto-type tail implies weak Pareto-type tail. Under a Pareto distribution, $x^{\alpha}\bar{F}_X(x) = x_{\min}^{\alpha}$ for every $x \geqslant x_{\min}$, so the tail constant is $k_X = x_{\min}^{\alpha}$, and the distinction matters for top-share calculations: the top-share identity used in Section 5 is exact only for the full Pareto distribution. A Pareto tail is Pareto-type, because every positive function converging to a positive constant is slowly varying, and the converse fails: $\bar{F}_X(x) = x^{-\alpha}\log x$ for large x is Pareto-type but not a Pareto tail. Every Pareto-type tail is weak Pareto-type with the same index, because $\log L(x)/\log x \to 0$ for every slowly varying L (Bingham et al., 1987, Proposition 1.3.6). The converse again fails: $\bar{F}_X(x) = (2+\sin(\log x))/(3x)$ for $x \geqslant 1$ is squeezed between $1/(3x)$ and $1/x$, so its logarithmic index is one, yet the ratio $\bar{F}_X(e^{\pi}x)/\bar{F}_X(x)$ oscillates, so $\bar{F}_X \notin RV_{-1}$. The weak class takes the Pareto-exponent criterion of Beare and Toda (2022, Theorem 1) as the definition. It coincides with the weak Pareto type of Gulisashvili (2012, Chapter 7) and is weaker than the weak Pareto law of Mandelbrot (1960, Section 1.2), which requires $\bar{F}_X(x) \sim kx^{-\alpha}$.

If X and Y have Pareto-type tails with $\alpha_X > \alpha_Y$, then the ratio $\bar{F}_X(x)/\bar{F}_Y(x)$ equals $x^{-(\alpha_X-\alpha_Y)}L_X(x)/L_Y(x)$, which tends to zero because every positive power dominates a slowly varying function (Bingham et al., 1987, Proposition 1.3.6). Within these classes we therefore describe the variable with the larger index as having the thinner tail.

3.2 Statement

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space on which all random variables are defined. Let $W \in \mathbb{R}_+$ denote stationary wealth, let S be an exogenous random element of a measurable state space $\mathbb{S}$, and let $C = f(W, S)$ denote stationary consumption, where $f : \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}_+$ is a measurable consumption policy. The results of this section use only the stated conditions on f and not the household problem behind them. Optimality enters in Section 4, which derives the bound from the income-fluctuation problem.

Assumption 3.1 (Positive linear lower bound)**.** There exist constants $a > 0$ and $w_0 \geqslant 0$ such that

$$f(w, s) \geqslant aw \tag{2}$$

for all $w \geqslant w_0$ and all $s \in \mathbb{S}$.

Remark 3.1. Assumption 3.1 is the key condition in the theorem. It says that beyond some wealth level, consumption never falls below a fixed fraction a of wealth. The lower bound need not hold globally (only eventually, for $w \geqslant w_0$) and the policy need not be linear.

Theorem 3.1 (Impossibility of a thinner consumption tail under a positive linear lower bound)**.** *Let W, S, C, f be defined as above and suppose W has a Pareto-type tail with index $\alpha_W > 0$. If Assumption 3.1 holds, then*

$$\liminf_{x\to\infty} \frac{\bar{F}_C(x)}{\bar{F}_W(x)} \geqslant a^{\alpha_W} > 0.$$

If, in addition, C has a Pareto-type tail with index α_C, then $\alpha_C \leqslant \alpha_W$.

Proof. Let a and w_0 satisfy Assumption 3.1. If $x \geqslant aw_0$ and $W > x/a$, then $W \geqslant w_0$ and $C = f(W, S) \geqslant aW > x$. Hence, for all $x \geqslant aw_0$, $\bar{F}_C(x) = \mathbb{P}\{C > x\} \geqslant \mathbb{P}\{W > x/a\} = \bar{F}_W(x/a)$. Since $\bar{F}_W \in RV_{-\alpha_W}$, $\bar{F}_W(x/a)/\bar{F}_W(x) \to a^{\alpha_W}$. This gives the liminf bound, so $\bar{F}_C(x)/\bar{F}_W(x)$ does not converge to zero. Finally, suppose C has a Pareto-type tail with index α_C, hence a weak Pareto-type tail with the same index (Subsection 3.1). The

same lower bound gives $-\log \bar{F}_C(x)/\log x \leqslant -\log \bar{F}_W(x/a)/\log x$ for all sufficiently large x. The left side tends to α_C, while the right side tends to α_W. Hence $\alpha_C \leqslant \alpha_W$. □

Remark 3.2 (A weaker tail-exposure condition)**.** The uniform policy bound is a transparent sufficient condition, not the weakest possible one. If, for some $a > 0$,

$$p := \liminf_{x\to\infty} \mathbb{P}\{C \geqslant aW \mid W > x\} > 0, \tag{3}$$

then

$$\liminf_{x\to\infty} \frac{\bar{F}_C(x)}{\bar{F}_W(x)} \geqslant p\, a^{\alpha_W} > 0,$$

and $\alpha_C \leqslant \alpha_W$ whenever C has a Pareto-type tail. The proof is unchanged apart from the constant: $\bar{F}_C(x) \geqslant p' \bar{F}_W(x/a)$ for every $p' \in (0, p)$ and all sufficiently large x, and a positive constant does not affect the logarithmic limits. An independent Bernoulli exposure with success probability p attains the smaller coefficient, so the constant is sharp in general. Assumption 3.1 implies (3) with conditional probability one.

3.3 Intuition: the linear case

A simple counterfactual makes the logic concrete. In the Australian data, the complementary CDF of nondurable consumption falls more steeply on logarithmic axes than that of net worth ($\widehat{\alpha}_C = 3.168$ against $\widehat{\alpha}_W = 2.119$, Table 1). A counterfactual rule $C = 0.5W$ shifts the wealth tail horizontally but does not change its slope. This is exactly what Lemma A.1 predicts at $\rho = 1$: changing the constant a shifts the counterfactual line but does not alter its tail index. A linear consumption rule forces $\alpha_C = \alpha_W$, ruling out the empirical ranking $\alpha_C > \alpha_W$. Extending the argument to state-dependent policies requires the conditions of Subsection 3.4.

Consider households in a Pareto-type wealth tail, regardless of how that tail is generated. If the policy keeps consumption above a fixed positive fraction of wealth at high wealth levels, then exceptionally wealthy households must also have exceptionally high consumption. The consumption tail is therefore at least as heavy as the wealth tail.

3.4 Sufficient conditions: affine and asymptotically linear policies

The sufficient conditions in this subsection use compactness of the state space and continuity of the policy to obtain a uniform positive coefficient.

Recall from Definition A.1(i) that a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is asymptotically linear if $\varphi(x)/x \to k$ for some constant $k > 0$. Here the definition is applied state by state, to $w \mapsto f(w, s)$ for each $s \in \mathbb{S}$.

Assumption 3.2 (State space)**.** The state space $\mathbb{S}$ is a compact metric space equipped with its Borel sigma-algebra.

Assumption 3.3 (Exact affine policy)**.** Suppose Assumption 3.2 holds. There exist continuous functions $h : \mathbb{S} \to (0, \infty)$ and $g : \mathbb{S} \to [0, \infty)$ such that $f(w, s) = h(s)w + g(s)$ for all $(w, s) \in \mathbb{R}_+ \times \mathbb{S}$.

Assumption 3.4 (Concave asymptotically linear policy)**.** Suppose Assumption 3.2 holds and f is continuous on $\mathbb{R}_+ \times \mathbb{S}$. For every $s \in \mathbb{S}$, the function $f(\cdot, s)$ is concave. There exists a continuous function $K : \mathbb{S} \to (0, \infty)$ such that

$$\lim_{w \to \infty} f(w, s)/w = K(s) \tag{4}$$

for all $s \in \mathbb{S}$.

Proposition 3.2. *Let f be defined as above. If f satisfies either Assumption 3.3 or Assumption 3.4, then f satisfies Assumption 3.1.*

Proof of Proposition 3.2. Case 1: Suppose f satisfies Assumption 3.3. Since h is continuous on the compact set $\mathbb{S}$ with $h : \mathbb{S} \to (0, \infty)$, the minimum $a := \min_{s \in \mathbb{S}} h(s)$ is positive. Then $f(w, s) = h(s)w + g(s) \geqslant aw$ for all $w \in \mathbb{R}_+$ and all $s \in \mathbb{S}$. Thus Assumption 3.1 holds with $w_0 = 0$.

Case 2: Suppose f satisfies Assumption 3.4. Fix $s \in \mathbb{S}$. Since $f(\cdot, s)$ is concave with $f(0, s) \geqslant 0$, the ratio $f(w, s)/w$ is nonincreasing in w on $(0, \infty)$: for $0 < w_1 < w_2$, write $w_1 = (w_1/w_2)\, w_2 + (1 - w_1/w_2) \cdot 0$, so concavity gives $f(w_1, s) \geqslant (w_1/w_2)\, f(w_2, s)$, hence $f(w_1, s)/w_1 \geqslant f(w_2, s)/w_2$. It follows that $f(w, s)/w \geqslant \lim_{w' \to \infty} f(w', s)/w' = K(s)$ for all $w > 0$. Since K is continuous on the compact set $\mathbb{S}$ with $K(s) > 0$, setting $a := \min_{s \in \mathbb{S}} K(s) > 0$ gives $f(w, s) \geqslant a\, w$ for all $w > 0$ and all $s \in \mathbb{S}$. Thus Assumption 3.1 holds with $w_0 = 0$. □

Remark 3.3. Both sufficient conditions deliver the global bound with $w_0 = 0$. This is stronger than the tail condition required by Theorem 3.1.

3.5 Exact tail index inheritance under independence

When wealth has a Pareto tail, not merely a Pareto-type tail, and is independent of the exogenous state, the affine and concave asymptotically linear policies of Subsection 3.4 transmit a Pareto tail with the same index and an explicit constant.

Proposition 3.3 (Exact tail inheritance)**.** *Let W, S, C be defined as above, and suppose W has a Pareto tail with index $\alpha_W > 0$ and constant k_W. If either Assumption 3.3 or 3.4 holds and W and S are independent, then C has a Pareto tail with the same index. Its constant is*

$$k_C = k_W \mathbb{E}[H(S)^{\alpha_W}],$$

where $H = h$ under Assumption 3.3 and $H = K$ under Assumption 3.4.

Sketch of the proof. Under Assumption 3.3, boundedness of g on compact $\mathbb{S}$ gives uniform convergence of $f(w,s)/w$ to $h(s)$. Under Assumption 3.4, concavity makes $f(w,s)/w$ decrease to $K(s)$. Dini's theorem (Rudin, 1976, Theorem 7.13) then makes the convergence uniform on $\mathbb{S}$. In either case, f has uniform asymptotic power growth with exponent one. Corollary A.4 gives the result. Appendix A supplies the uniformity details. □

Remark 3.4. Gaillard et al. (2023, Proposition 1) obtain $\alpha_C = \alpha_W$ in their continuous-time model with homothetic preferences and scale-independent returns, under their Proposition 1 conditions. Proposition 3.3 establishes the same equality under its policy conditions and independence, whereas Theorem 3.1 rules out $\alpha_C > \alpha_W$ under its stated assumptions, without independence or a tail-factorization condition. Appendix C compares these assumptions and states the open extension.

3.6 The escape criterion

Theorem 3.1 applies whenever wealth has a Pareto-type tail and consumption satisfies Assumption 3.1, so a model matching the empirical ranking $\alpha_C > \alpha_W$ must break the positive linear lower bound. Equivalently,

$$\liminf_{w\to\infty} \inf_{s\in\mathbb{S}} \frac{f(w,s)}{w} = 0,$$

which is weaker than requiring $f(w,s)/w \to 0$ in every state. The ranking also has a distributional counterpart.

Corollary 3.4 (Vanishing exposure in the wealth tail)**.** *If W and C have Pareto-type tails with $\alpha_C > \alpha_W$, then, for every $a > 0$,*

$$\mathbb{P}\{C > aW \mid W > x\} \longrightarrow 0 \qquad \text{as } x \to \infty. \tag{5}$$

Proof. Since $\{C > aW,\, W > x\} \subseteq \{C > ax\}$,

$$\mathbb{P}\{C > aW \mid W > x\} \leqslant \frac{\bar{F}_C(ax)}{\bar{F}_W(x)} = \frac{\bar{F}_C(ax)}{\bar{F}_C(x)} \cdot \frac{\bar{F}_C(x)}{\bar{F}_W(x)}.$$

The first factor converges to $a^{-\alpha_C}$ because $\bar{F}_C \in RV_{-\alpha_C}$, and the second tends to zero because $\alpha_C > \alpha_W$ (Subsection 3.1). □

Among increasingly wealthy households, the fraction consuming any fixed positive share of wealth must therefore vanish. This is necessary for the observed tail ranking, not sufficient to generate it: $C = W/\log(e + W)$ has a vanishing share but the same Pareto-type index as W. The paper uses this statement as a diagnostic rather than developing a structural escape mechanism.

One sufficient condition for escape is uniform asymptotic power growth with policy exponent $\rho \in (0, 1)$ in the sense of Definition A.1. Then $f(w, s)/w \to 0$ uniformly in s. If wealth has a Pareto-type tail and the remaining conditions of Theorem A.3 hold, then C has a weak Pareto-type tail (Definition 3.2(ii)) with index $\alpha_C = \alpha_W/\rho$. Corollary A.4 gives a Pareto tail under its stronger conditions. Under a stylized wealth-only selection model (Hu, 2026, Chapter 2), both observed indices are inflated but their ratio is unchanged. This invariance need not hold when selection depends on consumption or state, the scale factor is unbounded, or frame or measurement error is present.

4 The income-fluctuation model

The income-fluctuation model of Ma et al. (2020) extends the canonical Bewley–Huggett–Aiyagari environment to Markov-modulated discount factors, returns, and income. This section states the problem in full and shows that a stability condition on the savings rate (Assumption 4.2) forces the positive linear lower bound of Assumption 3.1 for any admissible utility (Ma et al., 2020, Proposition 2.6), so Theorem 3.1 applies.

Under CRRA utility, the same stability condition is among the conditions that generate heavy-tailed wealth through a Kesten-type argument (Ma et al., 2020, Theorem 3.3) (Kesten, 1973; Goldie, 1991), creating the Pareto trap (Theorem 4.2). Shifted HARA utility delivers the same bound under the alternative foundation of Ma and Toda (2022) (Corollary 4.4).

4.1 The income-fluctuation problem

We adopt the setup of Ma et al. (2020). Let $\mathbb{S} = \{s_1, \ldots, s_n\}$ be a finite exogenous state space and let $\{s_t\}$ be an irreducible Markov chain with transition matrix $Q =$

(Q_{ij}). Irreducibility gives the finite exogenous chain a unique stationary distribution and makes every state reachable from every other state. Each household has wealth $w_t \in \mathbb{R}_+$ and state $s_t \in \mathbb{S}$, and chooses $c_t \in [0, w_t]$. Set $\beta_0 := 1$. The objective is

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \left(\prod_{i=0}^{t} \beta_i \right) u(c_t) \tag{6}$$

subject to

$$w_{t+1} = R_{t+1}(w_t - c_t) + Y_{t+1}, \qquad 0 \leqslant c_t \leqslant w_t, \tag{7}$$

where $u : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ is the utility function. Here w_t is wealth available before consumption, so $w_t - c_t \geqslant 0$ is saving carried into the return process. The feasibility restriction therefore combines the timing convention of Ma et al. (2020) with a nonnegative-saving constraint. Some Bewley–Huggett–Aiyagari formulations instead use beginning-of-period assets and a separate lower bound on next-period assets, which may permit negative positions. The restriction $c_t \leqslant w_t$ enters the irreducibility and ergodicity arguments because the borrowing constraint must bind with positive probability (Ma et al., 2020, Lemmas C.2–C.3).

Remark 4.1. The constraint $c_t \leqslant w_t$ is a primitive of the environment, not a derived property. Under the conditions of Corollary 4.3 it cannot bind asymptotically: $m_\infty(s) < 1$ in every state, so consumption falls strictly inside the budget set at high wealth.

For $t \geqslant 1$, the discount factor, gross return, and nonfinancial income are

$$\beta_t := \beta(s_t, \varepsilon_t), \qquad R_t := R(s_t, \zeta_t), \qquad Y_t := Y(s_t, \eta_t),$$

for measurable functions $\beta : \mathbb{S} \times \mathbb{R} \to (0, \infty)$, $R : \mathbb{S} \times \mathbb{R} \to (0, \infty)$, and $Y : \mathbb{S} \times \mathbb{R} \to \mathbb{R}_+$. The innovation sequences are i.i.d. across time, mutually independent, and independent of $\{s_t\}$. Let $\mathbb{X} := \mathbb{R}_+ \times \mathbb{S}$. The initial condition $(w_0, s_0) \in \mathbb{X}$ is given, and $\mathbb{E}_0$ denotes expectation conditional on it.

4.1.1 Existence and uniqueness of the optimal policy

The following conditions ensure existence and uniqueness of an optimal policy (Ma et al., 2020, Sections 2.1–2.4):

Assumption 4.1 (Existence and uniqueness conditions). (i) **Utility.** u is twice differentiable on $(0, \infty)$, with $u' > 0$ and $u'' < 0$. Moreover, $\lim_{c \to 0^+} u'(c) = \infty$ and $\lim_{c \to \infty} u'(c) < 1$.

(ii) **Discounting.** $\lim_{n\to\infty}(\mathbb{E}\prod_{t=1}^{n}\beta_t)^{1/n} < 1$.

(iii) **Growth.** $\lim_{n\to\infty}(\mathbb{E}\prod_{t=1}^{n}\beta_t R_t)^{1/n} < 1$.

(iv) **Integrability.** $\mathbb{E}Y < \infty$ and $\mathbb{E}u'(Y) < \infty$.

Condition (i) imposes strict concavity, the Inada condition at zero, and the high-consumption bound on marginal utility used by Ma et al. (2020). The leading example is CRRA utility, $u(c) = c^{1-\gamma}/(1-\gamma)$ for $\gamma > 0$ and $\gamma \neq 1$, with $u(c) = \ln c$ when $\gamma = 1$. Condition (ii) requires geometric discounting on average. Condition (iii) bounds the long-run discounted return, preventing indefinite postponement of consumption. Condition (iv) requires integrability of income and of marginal utility evaluated at income.

Under Assumption 4.1, the time-iteration operator of Ma et al. (2020, Theorem 2.2) has a unique fixed point f in their class of continuous, wealth-increasing feasible policies, and this fixed point is the class's unique optimal policy. Extend it to zero wealth by setting $f(0,s) = 0$. Then $f : \mathbb{X} \to \mathbb{R}_+$ satisfies $0 \leqslant f(w,s) \leqslant w$. The optimal saving policy $g(w,s) := w - f(w,s)$ is also increasing in wealth (Ma et al., 2020, Proposition 2.3).

Write $\beta' := \beta(s',\varepsilon')$, $R' := R(s',\zeta')$, and $Y' := Y(s',\eta')$ for next-period variables, and let $\mathbb{E}_s$ denote expectation conditional on current state s. The value function satisfies the Bellman equation

$$V(w,s) = \max_{0\leqslant c\leqslant w}\left\{u(c) + \mathbb{E}_s\left[\beta' V\big(R'(w-c)+Y',s'\big)\right]\right\}. \tag{8}$$

At interior states where the relevant derivatives exist, the Euler equation takes the form

$$u'\big(f(w,s)\big) = \mathbb{E}_s\left[\beta' R' u'\big(f(R'(w-f(w,s))+Y',s')\big)\right], \tag{9}$$

and the envelope identity is

$$V_w(w,s) = u'\big(f(w,s)\big). \tag{10}$$

These equations are used only to verify the high-wealth shape of f.

4.1.2 The stability condition

The stability condition serves four roles: it forces optimal consumption to satisfy Assumption 3.1 (Proposition 4.1), it provides the savings upper bound for stationarity of the wealth distribution (Ma et al., 2020, Theorem 3.1), it supports uniqueness

and ergodicity of that distribution (Ma et al., 2020, Theorem 3.2), and it enters the conditions used to obtain a Pareto-order lower bound for wealth (Ma et al., 2020, Theorem 3.3).

Assumption 4.2 (Stability condition)**.** There exists $\bar{x} \in (0,1)$ such that, for every $w > 0$ and $i = 1, \ldots, n$,

$$\sum_{j=1}^{n} Q_{ij}\, \mathbb{E}\left[\beta(s_j, \varepsilon)\, R(s_j, \zeta)\, u'\big(R(s_j, \zeta)\, \bar{x}\, w\big)\right] \leqslant u'(w). \tag{11}$$

Conditional on current state s_i, the left-hand side of (11) is the expected marginal value of saving fraction $\bar{x}$ of wealth w when nonfinancial income is zero. The right-hand side is the marginal utility of consuming all wealth. The inequality says saving fraction $\bar{x}$ is suboptimal even in this zero-income case, forcing $f(w,s) \geqslant (1-\bar{x})w > 0$ for all $w > 0$ and $s \in \mathbb{S}$ (Proposition 4.1). This is Ma et al. (2020, condition (19)).

Remark 4.2 (Income and the stability condition)**.** Since $Y \geqslant 0$ and $u'' < 0$, adding income $Y(s_j, \eta)$ inside $u'(\cdot)$ in (11) reduces the left-hand side, so Assumption 4.2 implies the weaker condition with income included. We state the stronger form because it yields the linear lower bound for general utility (Proposition 4.1) and is one component of Ma et al. (2020, Assumption 3.1), which is used for stationarity and tail behavior.

4.2 The Pareto trap

The positive linear lower bound in Assumption 3.1 follows from the stability condition (Assumption 4.2) alone, for any utility satisfying Assumption 4.1. Under CRRA utility, the same constant $\bar{x}$ also enters the conditions for heavy-tailed stationary wealth: the conditions that generate heavy tails simultaneously lock in the lower bound.

Proposition 4.1 (From stability to the linear lower bound)**.** *If Assumptions 4.1 and 4.2 hold, then the optimal consumption f satisfies $f(w,s) \geqslant (1-\bar{x})w$ for all $w > 0$ and $s \in \mathbb{S}$. In particular, Assumption 3.1 holds with $a := 1 - \bar{x}$ and $w_0 = 0$.*

Proof. This is Ma et al. (2020, Proposition 2.6). Assumption 4.2 is their condition (19), and their conclusion is the displayed lower bound. □

By Theorem 3.1, whenever stationary wealth and consumption have Pareto-type tails and the stability condition holds, the ranking $\alpha_C > \alpha_W$ is ruled out. The question is whether stationary wealth can have a heavy tail in the first place, and whether the conditions that produce it are compatible with the stability condition.

Definition 4.1 (Asymptotic MPC and saving rate)**.** Let $m_\infty(s) := \lim_{w\to\infty} f(w,s)/w$ whenever the limit exists. Write $s_\infty(s) := 1 - m_\infty(s)$ for the asymptotic saving rate. In the notation of Assumption 3.4, $m_\infty(s) = K(s)$.

Remark 4.3 (The four roles of $\bar{x}$)**.** The constant $\bar{x}$ from Assumption 4.2 enters the Ma et al. (2020) framework at four levels:

(i) *Linear lower bound.* Assumption 4.2 alone gives $f(w,s) \geqslant (1-\bar{x})w$ for any u satisfying Assumption 4.1 (Proposition 4.1).

(ii) *Existence of a stationary distribution.* Combined with a contraction-on-average condition on returns, stated formally in Assumption 4.3(i), the savings bound prevents wealth from drifting to infinity (Ma et al., 2020, Theorem 3.1). This holds for any u satisfying Assumption 4.1.

(iii) *Uniqueness and ergodicity.* With an additional mixing condition on income (Assumption 4.3(ii)), the stationary distribution is unique and ergodic (Ma et al., 2020, Theorem 3.2), again for any u satisfying Assumption 4.1.

(iv) *Heavy-tailed wealth.* Under CRRA utility, the optimal saving function is asymptotically linear with rate $s_\infty(s)$ (Definition 4.1), making wealth follow an approximately affine recursion at high levels. The classical theory of such recursions (Kesten, 1973; Goldie, 1991) then supplies a Pareto-order lower bound for the wealth tail (Ma et al., 2020, Theorem 3.3). Its expansion matrix uses the actual asymptotic saving rate $s_\infty(s)$, while $\bar{x}$ enters through the stationarity condition required by the theorem. This is the only level at which CRRA is used.

If the time-iteration (Coleman) operator (Ma et al., 2020, Section 2), which maps a candidate consumption policy to the solution of the Euler equation, preserves concavity of the policy function in wealth, then by Ma et al. (2020, Proposition 2.5) the optimal consumption f is concave in w for each s and asymptotically linear (Definition A.1(i)): $m_\infty(s)$ exists with $m_\infty(s) \in [0,1]$. This concavity-preservation property holds for CRRA utility, as shown in Remark 2.1 and Appendix B of Ma et al. (2020). More general asymptotic-linearity results for regularly varying marginal utility, including HARA specifications, are available under additional conditions (Ma and Toda, 2022). The stability condition (Assumption 4.2) sharpens the lower end: since $f(w,s) \geqslant (1-\bar{x})w$ by Proposition 4.1, concavity gives $m_\infty(s) \geqslant 1-\bar{x} > 0$.

Because $\mathbb{S}$ is finite, the convergence defining $m_\infty(s)$ is uniform across states. Hence $(w - f(w,s))/w \to s_\infty(s)$ uniformly in s. Substitution into (7) gives the high-wealth affine approximation

$$W_{t+1} \approx A_{t+1}W_t + B_{t+1}, \qquad A_{t+1} := s_\infty(s_t)\, R_{t+1}, \quad B_{t+1} := Y_{t+1}. \tag{12}$$

For the corresponding exact affine benchmark, define $G_{ij}(\zeta) := s_\infty(s_i)R(s_j,\zeta)$ and the full-multiplier Kesten matrix (cf. its calibration implementation in Section 5)

$$K_\alpha(i,j) := Q_{ij}\,\mathbb{E}\left[G_{ij}(\zeta)^\alpha\right], \qquad i,j = 1,\dots,n. \tag{13}$$

Under the classical conditions for an exact Markov affine recursion, a positive solution to $r(K_\alpha) = 1$, where $r(\cdot)$ denotes spectral radius, characterizes its Pareto tail index (Kesten, 1973; Goldie, 1991; Roitershtein, 2007).

The nonlinear income-fluctuation problem requires a distinct object. Following Ma et al. (2020, Theorem 3.3), define the expansion-only matrix

$$K_\kappa^+(i,j) := Q_{ij}\,\mathbb{E}\left[\left(G_{ij}(\zeta)\,\mathbb{1}\{G_{ij}(\zeta) > 1\}\right)^\kappa\right] \tag{14}$$

and $\lambda_+(\kappa) := r(K_\kappa^+)$. Its crossing point is

$$\kappa_W := \inf\{\kappa > 0 : \lambda_+(\kappa) > 1\}.$$

This crossing point governs the Pareto-order lower bound established by their theorem.

The superscript distinguishes K_κ^+ from the full-multiplier matrix K_α in (13). Only the expansion events $G_{ij}(\zeta) > 1$ enter K_κ^+, and κ_W need not equal the exact-affine tail index associated with K_α.

The following conditions, adapted from Ma et al. (2020, Assumptions 3.1–3.3 and Theorem 3.3), ensure stationarity and give a Pareto-order lower bound when the stationary wealth distribution has unbounded support:

Assumption 4.3 (Conditions for the heavy-tail lower bound)**.** [1] Let $\bar{x} \in (0,1)$ be the constant from Assumption 4.2.

(i) **Contraction on average.** $\bar{x}\,\lim_{n\to\infty}(\mathbb{E}\prod_{t=1}^n R_t)^{1/n} < 1$.

(ii) **Income support.** There exists $\bar{s} \in \mathbb{S}$ with $Q_{\bar{s}\bar{s}} > 0$. Let $y_\ell \geqslant 0$ be the greatest lower bound of the support of $\{Y_t\}$. Either the support is finite and $\mathbb{P}(Y_t = y_\ell \mid s_t = \bar{s}) > 0$, or $\{Y_t\}$ admits a conditional density that is positive on $(y_\ell, y_\ell + \delta)$ at $\bar{s}$ for some $\delta > 0$.

(iii) **Spread.** For the same $\bar{s}$ as in (ii), $\mathbb{P}(s_\infty(\bar{s})\,R(\bar{s},\zeta) > 1) > 0$.

[1] Under the notational correspondence $\mathbb{S} = Z$, $Q = P$, $w = a$, $f = c^*$, and our $\bar{x}$ is their $\bar{s}$, with Ma et al. (2020): Assumptions 4.2 and 4.3(i) restate their Assumption 3.1, condition (ii) restates their Assumption 3.2, and conditions (iii)-(iv), together with the direct asymptotic-MPC condition imposed below, provide the properties used in their Assumption 3.3 and Theorem 3.3.

(iv) **Crossing.** There exists $\theta > 0$ such that $1 < \lambda_+(\theta) < \infty$.

Condition (i) uses the *same* $\bar{x}$ from the stability condition (Assumption 4.2): the savings upper bound, combined with the long-run return, contracts wealth on average. Condition (ii) ensures mixing (social mobility): currently rich households eventually receive low income and decline. Condition (iii) requires occasional multiplicative expansion: the effective return $s_\infty(\bar{s})\,R$ exceeds one with positive probability in at least one persistent exogenous state. Condition (iv) requires that this expansion pushes $\lambda_+(\kappa)$ above one at some finite κ.

If Assumptions 4.1, 4.2, and 4.3 hold and $m_\infty(s) \in (0,1)$ exists for every $s \in \mathbb{S}$ (Definition 4.1), then the arguments of Ma et al. (2020, Theorems 3.1–3.3) apply. Their Assumption 3.3 obtains the asymptotic MPC through Proposition 2.5. Here it is imposed directly. A unique stationary wealth distribution exists, and if it has unbounded support, then for every $\varepsilon > 0$,

$$\liminf_{w\to\infty} \; w^{\kappa_W+\varepsilon}\,\mathbb{P}(W > w) \;>\; 0. \tag{15}$$

Combining the linear lower bound from Proposition 4.1 with (15) yields the following model-specific impossibility result.

Theorem 4.2 (The Pareto trap)**.** *If Assumptions 4.1, 4.2, and 4.3 hold, $m_\infty(s) \in (0,1)$ exists for every $s \in \mathbb{S}$ (Definition 4.1), and stationary wealth W has unbounded support, then C cannot have a Pareto-type tail with index $\alpha_C > \kappa_W$.*

Proof. By Proposition 4.1, $C = f(W,S) \geqslant (1-\bar{x})W$ almost surely, where $\bar{x} \in (0,1)$ is the constant from Assumption 4.2. Hence for every $c > 0$,

$$\bar{F}_C(c) \;\geqslant\; \bar{F}_W\!\left(\frac{c}{1-\bar{x}}\right).$$

Suppose to the contrary that C has a Pareto-type tail with index $\alpha_C > \kappa_W$, so that $\bar{F}_C(c) = c^{-\alpha_C}L_C(c)$ for a slowly varying function L_C. By (15), for every $\varepsilon > 0$,

$$\liminf_{c\to\infty} \; c^{\kappa_W+\varepsilon}\,\bar{F}_C(c) \;\geqslant\; (1-\bar{x})^{\kappa_W+\varepsilon} \liminf_{c\to\infty} \; \left(\frac{c}{1-\bar{x}}\right)^{\kappa_W+\varepsilon} \bar{F}_W\!\left(\frac{c}{1-\bar{x}}\right) \;>\; 0.$$

Pick $\varepsilon > 0$ with $\kappa_W + \varepsilon < \alpha_C$. Then

$$c^{\kappa_W+\varepsilon}\,\bar{F}_C(c) \;=\; c^{-(\alpha_C-\kappa_W-\varepsilon)}L_C(c).$$

Because every positive power dominates a slowly varying function (Bingham et al.,

1987, Proposition 1.3.6), the right-hand side tends to zero. Hence

$$\lim_{c\to\infty} c^{\kappa_W+\varepsilon}\bar{F}_C(c) = 0,$$

contradicting the positive $\liminf$ above. □

Corollary 4.3 (The Pareto trap under CRRA)**.** *If u is CRRA with $\gamma > 0$ and Assumptions 4.1, 4.2, and 4.3 hold, then $m_\infty(s) \in (0,1)$ exists for every $s \in \mathbb{S}$, and Theorem 4.2 applies whenever stationary wealth has unbounded support.*

Proof. CRRA satisfies the concavity-preservation property (condition (17) of Ma et al. 2020, their Remark 2.1 and Appendix B). By Ma et al. (2020, Proposition 2.5), f is concave in w and $m_\infty(s) := \lim_{w\to\infty} f(w,s)/w$ exists with $m_\infty(s) \in [0,1]$.

Lower bound. Since $f(w,s) \geqslant (1-\bar{x})w$ (Proposition 4.1) with $\bar{x} \in (0,1)$, we have $f(w,s)/w \geqslant 1-\bar{x}$ for all $w > 0$. Taking $w \to \infty$ gives $m_\infty(s) \geqslant 1-\bar{x} > 0$.

Upper bound. Suppose $m_\infty(s_i) = 1$ for some $i \in \{1,\dots,n\}$. Since $f(\cdot, s_i)$ is concave with $f(0,s_i) \geqslant 0$, the ratio $f(w,s_i)/w$ is nonincreasing in w on $(0,\infty)$. The ratio never exceeds one (feasibility) and has limit one, so it equals one for all $w > 0$. Hence $f(w,s_i) = w$ for all $w > 0$, meaning the household chooses zero saving at state s_i. The optimality condition (Ma et al., 2020, equation (10)) at the corner $c = w$ requires

$$u'(w) \;\geqslant\; \sum_{j=1}^{n} Q_{ij}\,\mathbb{E}\left[\beta(s_j,\varepsilon)\,R(s_j,\zeta)\,u'\big(f(Y(s_j,\eta),s_j)\big)\right].$$

Since $f(Y(s_j,\eta),s_j) \leqslant Y(s_j,\eta)$ (feasibility) and $u'' < 0$, the right-hand side is bounded below by $\sum_j Q_{ij}\,\mathbb{E}[\beta(s_j,\varepsilon)\,R(s_j,\zeta)\,u'(Y(s_j,\eta))]$, which is strictly positive by $\beta, R, u' > 0$. Under CRRA, $u'(w) = w^{-\gamma} \to 0$ as $w \to \infty$, so there exists a finite wealth threshold beyond which the inequality fails. This contradicts optimality of f, hence $m_\infty(s) < 1$ for all $s \in \mathbb{S}$. □

The next result establishes the linear lower bound for a shifted-HARA case outside Assumption 4.1(i). Its finite marginal utility at zero requires the alternative policy foundation of Ma and Toda (2022).

Corollary 4.4 (Shifted HARA under an alternative foundation)**.** *Suppose $u'(c) = (c+\bar{c})^{-\gamma}$ with $\bar{c} > 0$ and $\gamma > 0$, the processes (β, R, Y) satisfy Assumption 2 of Ma and Toda (2022), and Assumption 4.2 holds for this u, with constant $\bar{x}$. Then the income-fluctuation problem admits a unique optimal consumption policy f, and $f(w,s) \geqslant (1-\bar{x})w$ for every $w > 0$ and $s \in \mathbb{S}$, so Assumption 3.1 holds with $a = 1-\bar{x}$ and $w_0 = 0$.*

If, in addition, a stationary distribution for this policy gives wealth and consumption Pareto-type tails, then $\alpha_C \leqslant \alpha_W$.

Proof. Shifted HARA marginal utility is positive, continuous, strictly decreasing, and vanishes at infinity, so Assumption 1 of Ma and Toda (2022) holds, and $u'(0) = \bar{c}^{-\gamma}$ is finite and admissible in their framework (their Lemma 2.1). Within their Assumption 2, the two marginal-utility moments are finite for this u because $u' \leqslant \bar{c}^{-\gamma}$ and the conditional moments of βR are finite, while the income moment $\mathbb{E}Y < \infty$ is maintained. Their Theorem 2.2(ii) delivers the unique optimal policy f as the limit in their marginal-utility metric of $T^n c_0$ for any initial candidate c_0, where T is the policy-iteration operator of their Lemma 2.1.

The remaining argument specializes the proof of their Theorem 2.7 to the constant fraction $1-\bar{x}$ and threshold zero. Let C^* be the set of candidate consumption functions c with $c(w,s) \geqslant (1-\bar{x})w$ for every $w > 0$ and $s \in \mathbb{S}$. The initial candidate $c_0(w,s) = w$ belongs to C^*. The class is invariant under T. Fix $c \in C^*$, $w > 0$, and $s_i \in \mathbb{S}$, and write $\xi = Tc(w,s_i)$. Suppose $\xi < (1-\bar{x})w$. Then $w - \xi > \bar{x}w$, so every realized next-period wealth satisfies $R(s_j,\zeta)(w-\xi)+Y(s_j,\eta) \geqslant R(s_j,\zeta)\,\bar{x}\,w$, and the bound defining C^* gives $u'\big(c(R(s_j,\zeta)(w-\xi)+Y(s_j,\eta),s_j)\big) \leqslant u'\big(R(s_j,\zeta)\,\bar{x}\,(1-\bar{x})\,w\big)$. The Euler update of their Lemma 2.1 therefore yields

$$u'(\xi) \leqslant \max\Big\{\sum_{j=1}^{n} Q_{ij}\,\mathbb{E}\big[\beta(s_j,\varepsilon)\,R(s_j,\zeta)\,u'\big(R(s_j,\zeta)\bar{x}(1-\bar{x})w\big)\big],\, u'(w)\Big\} \leqslant u'\big((1-\bar{x})w\big),$$

where the sum is the left-hand side of Assumption 4.2 evaluated at wealth $(1-\bar{x})w$ and $u'(w) \leqslant u'((1-\bar{x})w)$ directly. Hence $\xi \geqslant (1-\bar{x})w$, a contradiction, so $Tc(w,s_i) \geqslant (1-\bar{x})w$. By induction $T^n c_0 \in C^*$ for every n, convergence in the marginal-utility metric implies pointwise convergence, and the bound passes to the limit, so $f(w,s) \geqslant (1-\bar{x})w$. The tail restriction follows from Theorem 3.1 with $a = 1-\bar{x}$ and $w_0 = 0$. □

Remark 4.4 (Scope of the shifted-HARA extension)**.** The corollary establishes the policy bound. Its tail-index implication is conditional on a stationary distribution with the stated Pareto-type tails. Neither existence of that distribution nor construction of the wealth tail is established here. At $\bar{c} = 0$ the utility is CRRA and the stronger Corollary 4.3 applies.

The optimal policy satisfies $0 \leqslant f(w,s) \leqslant w$, so the stationary pair satisfies $\mathbb{P}\{C \leqslant W\} = 1$: the model's global feasibility bound verifies the almost-sure assumption of the next corollary.

Corollary 4.5 (Feasibility and two-sided inheritance)**.** *Suppose Assumption 3.1 holds and* $\mathbb{P}\{C \leqslant W\} = 1$*. If* W *has a Pareto-type tail with index* α_W*, then* C *has a weak Pareto-type tail with index* α_W*.*

Proof. $\mathbb{P}\{C \leqslant W\} = 1$ gives eventual tail domination (Lemma A.5). Apply Corollary A.6 with Assumption 3.1. □

Thus the weak tail index is inherited without independence. Independence remains relevant to the stronger Pareto-tail conclusion and tail constant in Proposition 3.3.

Remark 4.5 (Tail bounds vs. Pareto tails)**.** Equation (15) is a $\liminf$ bound, not an exact tail. For an exact affine recursion, the classical theory can deliver a Pareto tail under its additional moment, irreducibility, and nonarithmetic conditions (Kesten, 1973; Goldie, 1991; Buraczewski et al., 2016). The root of the full-multiplier matrix K_α is the corresponding exact-affine tail index, while κ_W comes from the expansion-only matrix K_κ^+ used in the nonlinear lower-bound argument. If the stationary wealth distribution itself has a Pareto-type index α_W, Theorem 3.1 gives the sharper and directly comparable restriction $\alpha_C \leqslant \alpha_W$.

Remark 4.6 (Scope)**.** Subject to its remaining assumptions, Theorem 4.2 applies to any utility under which $m_\infty(s) \in (0,1)$ exists in every state. Corollary 4.3 covers CRRA specifications within the Ma et al. (2020) framework whenever the stated assumptions hold, and Corollary 4.4 provides the shifted-HARA policy bound under the alternative foundation of Ma and Toda (2022). Appendix B classifies related heterogeneous-agent models (Benhabib et al., 2011, 2015, 2016; Toda, 2019; Hubmer et al., 2021; Kaymak and Poschke, 2016) against the theorem's conditions.

Definition 4.2 (Stationary recursive competitive equilibrium)**.** Given primitives, a stationary recursive competitive equilibrium consists of a price vector, a household policy f, and a distribution φ. At those prices, f solves the household problem, φ is invariant under the Markov kernel induced by f and the exogenous state process, asset and goods markets clear, and prices satisfy firm optimality.

Remark 4.7 (General equilibrium)**.** The impossibility is proved conditional on a price vector. In a stationary recursive competitive equilibrium (Definition 4.2), prices are selected so that the household problem generates an invariant distribution consistent with market clearing. At those prices, the household problem is the income-fluctuation problem (6)-(7), so Theorem 4.2 applies whenever Assumptions 4.1, 4.2, and 4.3 hold there, $m_\infty(s) \in (0,1)$ exists in every state, and stationary wealth has unbounded support. The stability condition is part of the sufficient conditions for

stationarity in Ma et al. (2020, Theorem 3.1), not a necessary condition for every stationary equilibrium. General equilibrium can therefore affect whether the hypotheses of the Pareto trap hold, but it does not alter the conditional tail implication once they do.

4.3 A second application: finitely lived households

The income-fluctuation application uses the cash-on-hand bound $C \leqslant W$. The Pareto trap does not depend on that feature. In the finitely lived overlapping-generations economy of Benhabib et al. (2011), the optimal consumption rule at age τ is $C = m(\tau, R)\{W + H(\tau, R, Y)\}$, where H is remaining human wealth (Benhabib et al., 2011, Appendix A), so consumption can exceed current financial wealth.

Proposition 4.6 (OLG population tails with human wealth)**.** *Impose Assumptions 1–3 and the regularity conditions of Theorems 1–2 of Benhabib et al. (2011). In their stationary population cross-section, financial wealth satisfies the two-sided power-tail bound*

$$0 < \liminf_{v\to\infty} v^{\mu}\bar{F}_W(v) \leqslant \limsup_{v\to\infty} v^{\mu}\bar{F}_W(v) < \infty$$

for some $\mu > 1$, and consumption satisfies the same two-sided bound with the same exponent. Hence W and C have weak Pareto-type tails with index μ, and if C has a Pareto tail, its index is μ.

Proof. Their Lemma A.2 gives a unique ergodic dynasty process, their Theorem 1 gives an exact Pareto tail for the initial-wealth law, and their Theorem 2 transfers the power exponent μ to the stationary population wealth distribution in the stated two-sided form. For each of the finitely many return states, the propensity $m(\tau, R)$ is positive and continuous on the compact age interval, so across ages and return states it attains a positive minimum m_- and a finite maximum m_+, and with finitely many income states remaining human wealth is bounded by some $H_+ < \infty$. The affine rule then gives

$$m_- W \;\leqslant\; C \;\leqslant\; m_+ W + m_+ H_+ \qquad \text{almost surely.}$$

For $x > 0$, $\{W > x/m_-\} \subseteq \{C > x\}$, so $\bar{F}_C(x) \geqslant \bar{F}_W(x/m_-)$, and for $x > m_+H_+$, $\{C > x\} \subseteq \{W > (x - m_+H_+)/m_+\}$, so $\bar{F}_C(x) \leqslant \bar{F}_W((x - m_+H_+)/m_+)$. Both rescalings preserve two-sided v^{μ} bounds: $x^{\mu}\bar{F}_W(x/m_-) = m_-^{\mu}(x/m_-)^{\mu}\bar{F}_W(x/m_-)$, and $(x - m_+H_+)/m_+ \sim x/m_+$ makes the upper comparison asymptotically a fixed rescaling. The two-sided bound for C follows. The two-sided bounds give the logarithmic limits, so both weak Pareto-type indices equal μ, and a Pareto tail for C pins its index at its weak index. □

Thus the Pareto trap survives $C > W$ when the extra resource is bounded, with the model's own ergodicity and power-tail theorems supplying the stationary distribution. Their Theorem 1 concerns the initial-wealth law and their Theorem 2 the stationary population law, and their Claim 1 obtains its exact population Pareto constant under the assumption of i.i.d. returns.

4.4 Escape mechanisms

Subsection 3.6 states the necessary escape criterion: the positive linear lower bound must fail. This paper states the criterion and does not develop escape mechanisms. One published example satisfies it: under power wealth-in-utility preferences with $\delta < \gamma$, Ma and Toda (2025, Theorem 2) establish, under their additional conditions, including irreducibility, a spectral-radius restriction, an income floor, and positive probability of a positive discounted return from every state, that $f(w, s)$ grows like $w^{\delta/\gamma}$, which is uniform asymptotic power growth with exponent $\rho = \delta/\gamma < 1$ and, for wealth with a weak Pareto-type index α_W, maps to the weak consumption index $\alpha_C = \alpha_W \gamma/\delta$ through Theorem A.3 under its local-boundedness condition. When $\delta \geqslant \gamma$, the asymptotic MPC is strictly positive under the respective conditions of Ma and Toda (2025, Theorems 3–4), and the trap logic applies under the remaining hypotheses of Theorem 4.2. Hu (2026, Chapter 5) develops the wealth-motive and liquidation-cost mechanisms. In a separate return-risk example, Ma and Toda (2021) obtain a zero asymptotic MPC at sufficiently high risk aversion. Appendix B records the open model classes without assigning them an escape status. Section 5 quantifies the trap in the calibrated benchmark.

5 Calibration: the trap bites

This section calibrates the benchmark of Section 4 to each eligible wealth-tail index, quantifies the resulting implications for consumption, and states which empirical comparisons permit each calculation. Every empirical input comes from the prespecified release of Section 2, and the calibration carries forward the empirical covariance matrix, multiplicity decision, and object-specific eligibility recorded there. Index-calibration eligibility does not permit a top-share calculation under a Pareto distribution.

5.1 Calibration identities

The calibration runs from the wealth index to the return mean, inverting the asymptotic theory of Section 4: with lognormal returns, fixed preferences, and fixed return dispersion, the wealth target pins down the mean log return. Specialize to i.i.d. returns and income, a constant discount factor $\beta \in (0,1)$, and CRRA utility with $\gamma > 0$, including log utility, $\gamma = 1$. Maintain an interior asymptotic policy limit: as cash on hand grows, the fraction consumed approaches $m_\infty \in (0,1)$. The asymptotic saving rate then solves the high-wealth Euler identity

$$1 = \beta s_\infty^{-\gamma}\, \mathbb{E}[R^{1-\gamma}], \qquad s_\infty = \left(\beta\, \mathbb{E}[R^{1-\gamma}]\right)^{1/\gamma}, \tag{16}$$

obtained by substituting $c_t \approx m_\infty w_t$ and $c_{t+1} \approx m_\infty s_\infty R\, w_t$ into the Euler equation and canceling the common factor. The saving rate is the complementary share, $s_\infty = 1 - m_\infty$, and in the log case the identity gives $s_\infty = \beta$. For the corresponding exact affine recursion, the positive Kesten root solves

$$\mathbb{E}[A^\alpha] = \mathbb{E}[(s_\infty R)^\alpha] = 1, \tag{17}$$

the scalar case of the spectral condition $r(K_\alpha) = 1$ in equation (13). For lognormal returns, $\log R \sim N(\mu_R, \sigma_R^2)$ with μ_R the mean log return, distinct from the mean gross return $\mathbb{E}[R] = \exp(\mu_R + \sigma_R^2/2)$, the two identities solve in closed form for the mean log return that places the analytical root at a target α_W:

$$\mu_R = -\ln\beta - \frac{1}{2}(1-\gamma)^2\sigma_R^2 - \frac{1}{2}\gamma\,\alpha_W\,\sigma_R^2. \tag{18}$$

Thus $(\beta, \gamma, \sigma_R, \alpha_W)$ pin down μ_R analytically. The identities and the verification protocol follow Hu (2026, Chapter 2). Given each eligible wealth index, equation (18) therefore supplies the calibrated return mean directly. The calibration targets wealth alone, and the model's consumption tail is an untargeted prediction throughout.

5.2 Setup and calibration strategy

We calibrate the standard income-fluctuation problem of Section 4 to each eligible wealth-tail index and examine its untargeted implication for consumption. The empirical inputs are Australia (HILDA wave 22) and Germany, Italy, and the Netherlands (HFCS wave 5). Table 2 records their estimates and reporting status. The model restriction is asymptotic: under the maintained tail conditions, the positive linear lower

bound rules out $\alpha_C > \alpha_W$, while asymptotic linearity and independence sharpen the prediction to $\alpha_C = \alpha_W$.

The benchmark has CRRA utility $u(c) = c^{1-\gamma}/(1-\gamma)$ and i.i.d. lognormal shocks,

$$\log R \sim N(\mu_R, \sigma_R^2), \qquad \log Y \sim N(\mu_Y, \sigma_Y^2).$$

We set $\gamma = 1.5$, $\beta = 0.95$, $\sigma_R = 0.20$, $\sigma_Y = 0.50$, and normalize mean income to one. These are maintained illustrative values, not a joint structural estimate of preferences and both shock processes. For each eligible α_W, the code calibrates μ_R using the closed form (18) and imposes the scalar Kesten condition (17), where s_∞ is the analytical asymptotic saving rate of equation (16). It then solves the nonlinear problem by the endogenous grid method (Carroll, 2006) on a 600-point log grid. The stopping rule is a scale-adjusted sup norm of 10^{-11}. Write $s_{\text{term}} = 1 - f(w_{\max}^{\text{policy}})/w_{\max}^{\text{policy}}$ for the saving rate at the last policy-grid point. Separate checks cover the maximum relative residual of the interior Euler equation (9) (at most 10^{-3}), feasibility ($0 < f(w) \le w + 10^{-10}$), monotonicity (decreases of f at most 10^{-10}), contraction ($\mathbb{E}[\log(s_{\text{term}}R)] < 0$), the Kesten moment evaluated with s_{term} at the target (within 10^{-3} of one), and the positive root implied by s_{term} (within 10^{-2} of α_W).

Table 2: Empirical targets and object-specific reporting status

Country	$\hat{\alpha}_W$	$\hat{\alpha}_C$	Gap (s.e.)	Holm p	Reject	Exact tail
Australia	2.119	3.168	1.050 (0.199)	< 0.001	yes	no
Germany	1.603	4.085	2.482 (0.276)	0.002	yes	yes
Italy	1.615	3.383	1.768 (0.223)	< 0.001	yes	yes
Netherlands	2.096	2.906	0.810 (0.271)	0.081	no	no

All rows meet the index-calibration requirements. The Holm decision concerns the one-sided gap in the fixed weighted-survey family and is distinct from index eligibility. "Exact tail" permits pure-Pareto exponent benchmarks only. It does not identify a population top share without below-threshold aggregation.

5.3 Policy diagnostics

Table 3 reports the calibrated mean log return and the policy diagnostics. The consumption and saving shares in the table come from the last policy-grid point and agree with the analytical limits at the displayed precision. The last column evaluates $\mathbb{E}[(s_{\text{term}}R)^{\alpha_W}]$ and rounds to one in all four cases. All four calibrations pass the pre-

specified numerical checks. Equation (16) gives a limiting saving rate below one in each case. Proposition 4.1 supplies the global lower bound used by Theorem 3.1.

Table 3: Calibrated standard income-fluctuation parameters and policy diagnostics

Country	α_W	μ_R	$\mathbb{E}[R]$	m_∞	s_∞	$\lambda_K(\alpha_W)$
Australia	2.119	-0.017	1.003	0.025	0.975	1.000
Germany	1.603	-0.002	1.018	0.030	0.970	1.000
Italy	1.615	-0.002	1.018	0.030	0.970	1.000
Netherlands	2.096	-0.017	1.003	0.025	0.975	1.000

Targets are the four index-calibration-eligible units in the prespecified empirical release. Maintained parameters are $\gamma = 1.5$, $\beta = 0.95$, $\sigma_R = 0.20$, and $\sigma_Y = 0.50$. Eligibility is conditional on the empirical index extrapolation. It does not imply exact-tail eligibility.

Figure 2 displays the solved policies. The left panel plots $f(w)/w$ and its analytical limit. The right panel normalizes by that limit. Convergence is slow: at one hundred times mean income the consumption share is still more than twice its asymptote in every calibration, and at ten thousand times mean income it remains about 9 to 19 percent above the limit. This distinction between a finite policy profile and its limiting slope is central to interpreting the simulations below.

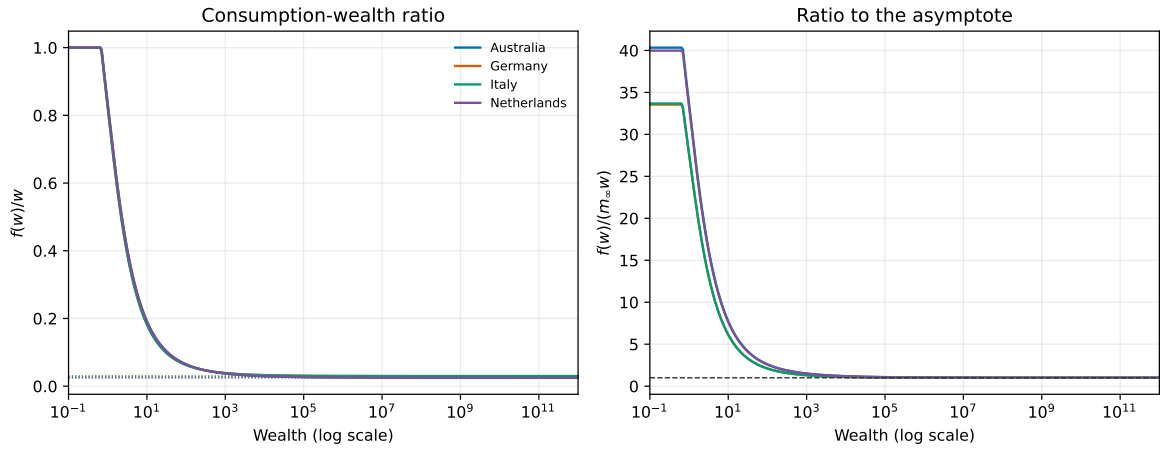

Figure 2: Policy diagnostics for the four index-calibration-eligible units. Dotted lines in the left panel mark the analytical asymptotic consumption-wealth ratios. The right panel shows the ratio to those limits.

5.4 Exact-affine tail diagnostic

To isolate the limiting multiplicative mechanism, we simulate the auxiliary recursion

$$W_{t+1} = s_{\text{term}} R_{t+1} W_t + Y_{t+1},$$

the numerical counterpart of the scalar high-wealth approximation in equation (12), and evaluate the solved nonlinear consumption policy on the resulting wealth cross-section. This calculation uses $N = 500{,}000$ agents, $T = 8{,}000$ periods, a common seeded shock stream, and no empirical unit records. It is an exact-affine tail diagnostic, not the full nonlinear stationary economy.

Table 4 reports fixed-fraction Hill estimates at the top 1%. The simulated wealth index is close to its Kesten target, while the consumption-to-wealth index ratio remains above one at this finite threshold. Figure 3 shows that the ratio moves toward one as the threshold advances further into the simulated tail. This pattern is consistent with exact asymptotic inheritance, but the finite simulation does not prove the limit.

Table 4: Exact-affine simulation diagnostics at the 1% tail fraction

Country	Target α_W	Sim. $\hat{\alpha}_W$	Sim. $\hat{\alpha}_C$	Ratio	Top 1% C	Gini(C)	Var($\ln C$)
Australia	2.119	2.056	2.585	1.258	6.0%	0.266	0.184
Germany	1.603	1.559	1.757	1.127	13.0%	0.393	0.333
Italy	1.615	1.573	1.777	1.130	12.7%	0.389	0.327
Netherlands	2.096	2.032	2.544	1.252	6.2%	0.270	0.188

The exact-affine high-wealth approximation uses $N = 500{,}000$ and $T = 8{,}000$ with a common seeded shock stream. Hill estimates use a fixed 1% sample fraction. Simulated top shares and body moments are model diagnostics, not empirical population shares. Ratios above one at this threshold are consistent with finite-range policy convergence and move toward one at deeper reported fractions. The full profile is shown in Figure 3.

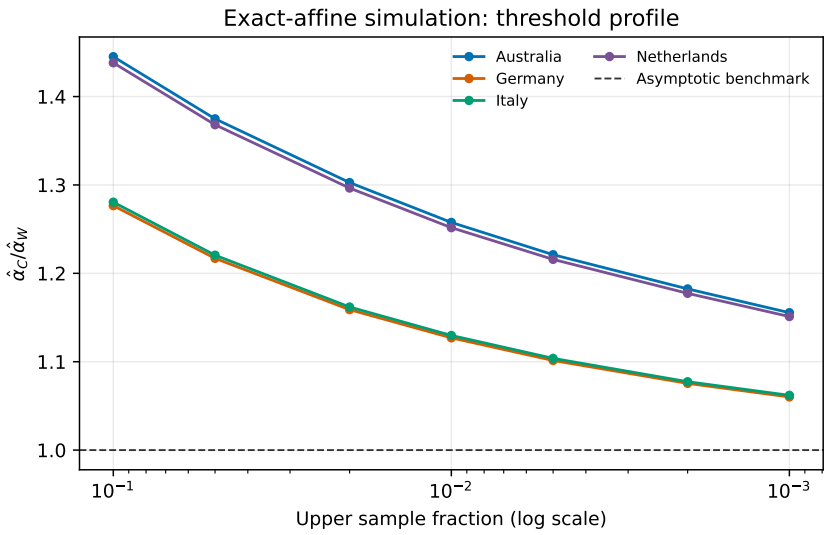

Figure 3: Consumption-to-wealth Hill-index ratios in the exact-affine auxiliary simulations. The horizontal line is the asymptotic benchmark. Moving right uses a smaller fraction of the sample further into the tail.

5.5 Full nonlinear stationary distribution

The affine comparison applies the terminal saving rate at every wealth level as a numerical approximation to the limiting rule. To measure the resulting approximation error, we also iterate the stationary distribution under the full Australian nonlinear policy. The comparison uses the same shocks and quadrature rule for both processes. Table 5 and Figure 4 report the primary grid. Appendix E describes the calculation and repeats it over three upper wealth bounds.

Table 5: Finite-grid distribution diagnostics for the Australian calibration

Process	Mean W	Gini(W)	Top 1% W	Gini(C)	Top 1% C	$\alpha_W(1\%)$	$\alpha_W(0.01\%)$
Full nonlinear policy	1.796	0.284	3.4%	0.116	1.6%	5.411	6.054
Exact-affine tail	45.194	0.475	12.3%	0.266	6.0%	2.004	2.106

The full process iterates the calibrated nonlinear policy. The exact-affine process imposes its limiting saving rate at every wealth level. The primary grid runs from 0.01 to $1{,}000{,}000$. All entries are model diagnostics. Local exponents describe the indicated finite tail fractions and do not establish asymptotic convergence.

Using s_{term} at every wealth level raises mean cash on hand from 1.8 to 45.2 times

mean income and lowers the local wealth-tail index at the top 1% from 5.41 to 2.00. Thus the exact-affine recursion changes body inequality and the onset of the tail even though it preserves the limiting Kesten mechanism. This is a finite-range finding, not a contradiction of the asymptotic theorem.

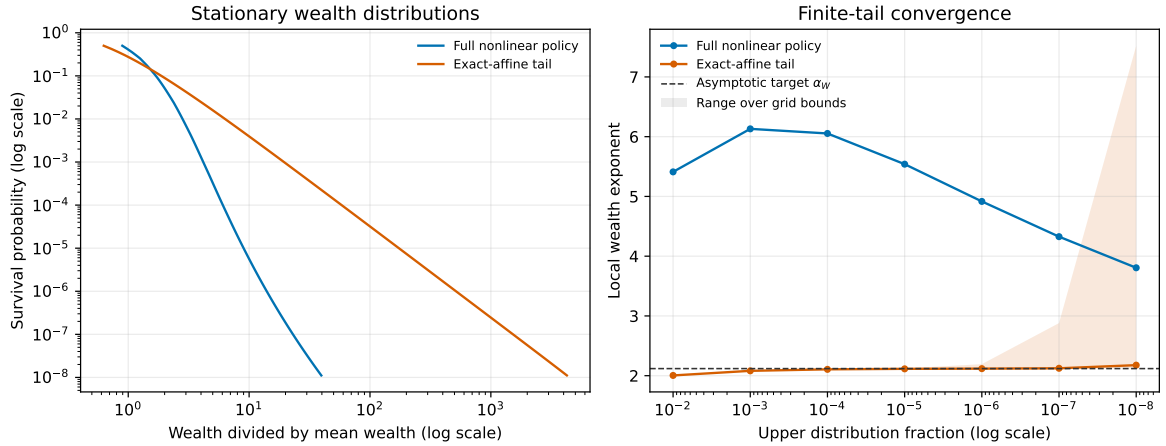

Figure 4: Australian stationary-distribution diagnostics. The left panel plots survival probability against cash on hand divided by its mean for each process. The right panel plots the local cash-on-hand exponent against survival probability. The curves use the primary 10^6 grid. Shading in the right panel spans the three grid ceilings, and the dashed line marks the calibrated Kesten target. The calculation describes behavior on these finite grids.

5.6 The impossibility scorecard

Figure 5 compares the four eligible index pairs with the i.i.d. asymptotic restriction $\alpha_C = \alpha_W$. All point estimates lie above the diagonal. In the fixed weighted-survey family, the one-sided Holm tests reject a nonpositive fitted consumption-minus-wealth gap for Australia, Germany, and Italy. The Dutch test does not reject. Under the maintained extrapolation to the limiting indices, the three rejections provide evidence against the benchmark restriction. Only Germany and Italy pass the separate screen for pure-Pareto benchmarks. An empirical population top share would also require the contribution below the fitted threshold.

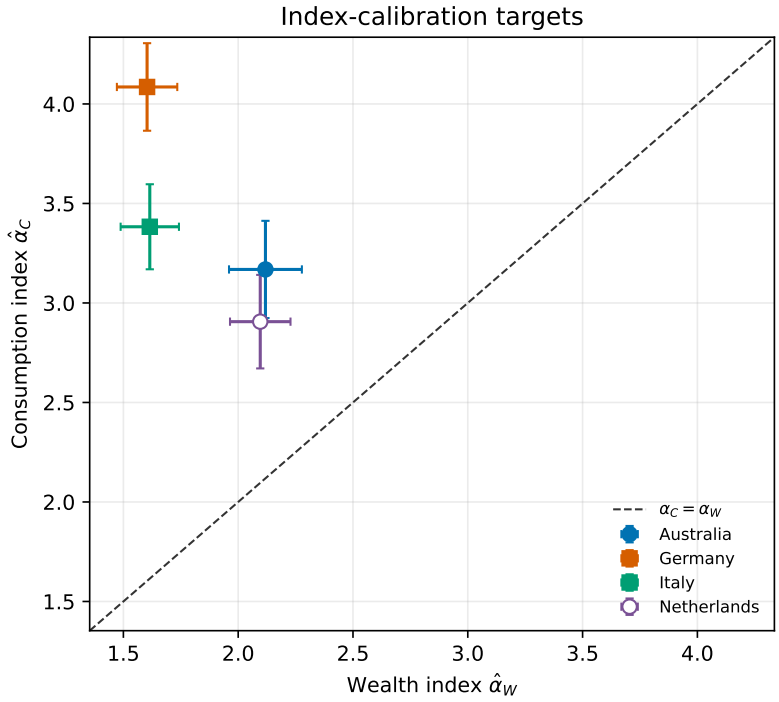

Figure 5: Index-calibration targets. Filled markers reject the one-sided gap null after Holm adjustment. Square markers pass the separate exact-tail benchmark. Error bars are marginal standard errors and do not replace the joint gap inference.

5.7 Robustness and scope

Return volatility. Table 6 holds the Australian wealth-index target, preferences, and income process fixed while varying σ_R. The return mean is recalibrated in every row and the same numerical checks are applied. Greater volatility requires a lower mean return and a smaller asymptotic consumption share to preserve the target tail index. This is a controlled sensitivity, not a structural estimate of the return process.

Table 6: Australian return-volatility sensitivity

σ_R	μ_R	$\mathbb{E}[R]$	m_∞	s_∞	$\lambda_K(\alpha_W)$	Euler residual
0.15	0.013	1.024	0.036	0.964	1.000	0.00011
0.20	-0.017	1.003	0.025	0.975	1.000	0.00013
0.25	-0.056	0.976	0.010	0.990	0.999	0.00011

Each row holds the wealth-index target, preferences, and income process fixed and recalibrates only the log-return mean. The extended policy grid is used for every row and all prespecified numerical checks pass. This controlled sensitivity is not a structural estimate of the return process.

Markov dependence. Theorem 3.1 permits arbitrary dependence between wealth and the finite Markov state and gives the one-sided restriction whenever the existence, stability, lower-bound, and tail assumptions hold. Exact inheritance requires the stronger conditions stated in Proposition 3.3 and is not asserted for a generic Markov specification. No finite Markov calibration is reported because the transition parameters are not identified from the tail estimates.

General equilibrium. The calibration is partial equilibrium. In a stationary recursive competitive equilibrium, prices are endogenous but fixed at their equilibrium values. The theorem applies at such a price vector only if its household-level assumptions hold there. General equilibrium can change both the validity of the stability condition and the numerical tail index. The condition must therefore be checked at the equilibrium prices, not assumed.

5.8 Implications for model design

The argument has four levels. First, Section 2 identifies finite-window index pairs and their uncertainty. Second, the calibration interprets eligible pairs as asymptotic targets under a maintained tail extrapolation. Third, the theorem rules out those targets in the positive-lower-bound class when its assumptions hold. Finally, exact-tail top-share calculations require the additional exact-tail benchmark. Keeping these levels separate prevents finite-window evidence from being reported as an assumption-free asymptotic rejection.

For a uniformly locally bounded policy with uniform asymptotic power growth as in Definition A.1, Theorem A.3 gives $\rho = \alpha_W/\alpha_C$ for the weak Pareto-type indices when

wealth has such a tail. Under the maintained extrapolation, the released estimates of this ratio range from 0.392 for Germany to 0.722 for the Netherlands. They serve as targets for the escape mechanisms in Hu (2026, Chapter 5), conditional on each mechanism's own eligibility requirements.

6 Conclusion

This paper treats the gap between consumption and wealth tails as a specification restriction for heterogeneous-agent models. The fitted consumption index exceeds the fitted wealth index in all seven weighted-survey comparisons of Section 2, consistent with existing U.S. evidence, and the difference survives the prespecified Holm adjustment for Australia, Germany, and Italy. The theoretical restriction does not depend on those particular estimates.

The main result follows from a single property of the consumption policy. If consumption remains above a positive fraction of resources at the top, the consumption tail cannot be strictly thinner than the resource tail. No upper feasibility bound or independence is needed for this one-sided impossibility. Feasibility sharpens the conclusion to weak tail-index inheritance without independence whenever wealth has a Pareto-type tail. Separately, with an exact Pareto wealth tail, independence and the affine or concave asymptotically linear policy conditions of Proposition 3.3 yield a Pareto consumption tail with an explicit constant. These sharpenings answer different questions and neither narrows the main theorem to a particular household budget.

The income-fluctuation application identifies the economic engine. The stability condition of Ma et al. (2020) supplies the difficult lower bound, while cash-on-hand feasibility supplies the upper bound. The same stability certificate also enters their stationarity and heavy-tail argument, creating the Pareto trap. In the finitely lived OLG economy of Benhabib et al. (2011), optimal consumption is affine in wealth, the same lower bound applies, and the model's own population power-tail bounds transfer to consumption (Proposition 4.6). Multi-asset and HANK models remain open here unless their lower exposure is established rather than inferred by analogy.

The full-Pareto example in the introduction illustrates the possible scale: replacing an index of 3.0 by 1.5 raises the analytical top 1% benchmark from 4.6% to 21.5%. The broader lesson is the overidentifying restriction. Once a model is calibrated to the wealth tail, its consumption tail is an untargeted validation outcome. A model that matches wealth but violates the consumption ranking has not matched the joint inequality facts it was asked to explain.

Declaration of generative AI and AI-assisted technologies in the writing process

During the late-stage preparation of this paper I used generative AI tools, principally Anthropic's Claude and OpenAI's Codex, working under my instruction at every step. The research questions, the theoretical results, and most of the manuscript predate my adoption of these tools. The tools polished the writing, checked statements, proofs, and cross-references, and helped develop and debug the replication code I wrote. I reviewed and checked every output line by line, independently verified all mathematical results, and take full responsibility for the content of this paper.

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A Tail mapping results and proofs

This appendix collects the general mapping results used in Sections 3 and 4: the asymptotic growth classes, the power-transformation and sandwich lemmas, the weak tail-mapping theorem, the Pareto tail constant, and the tail-domination results behind two-sided inheritance. It closes with the uniformity details for Proposition 3.3. Throughout, W, S, and $C = f(W, S)$ are as in Section 3.

Growth classes and regularity

Definition A.1 (Asymptotic growth classes)**.** Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be measurable. All limits in parts (i)-(iv) are taken as $x \to \infty$.

(i) g is *asymptotically linear* if

$$\frac{g(x)}{x} \to k \qquad \text{for some } k > 0.$$

(ii) g is *asymptotically sublinear* if

$$\frac{g(x)}{x} \to 0.$$

(iii) g has *asymptotic power growth* with exponent $\rho > 0$ if

$$g(x) \sim hx^{\rho} \qquad \text{for some scalar } h > 0.$$

(iv) A measurable function $f : \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}_+$ has *uniform asymptotic power growth* with exponent $\rho > 0$ if there is a function $H : \mathbb{S} \to (0, \infty)$ such that

$$\sup_{s \in \mathbb{S}} \left| \frac{f(w, s)}{w^{\rho}} - H(s) \right| \to 0 \qquad \text{as } w \to \infty,$$

where

$$0 < \inf_{s \in \mathbb{S}} H(s) \leqslant \sup_{s \in \mathbb{S}} H(s) < \infty.$$

For finite $\mathbb{S}$, the displayed bounds in part (iv) follow automatically from $H(s) \in (0, \infty)$ for every s. Power growth with $\rho = 1$ is equivalent to asymptotic linearity, and $0 < \rho < 1$ implies asymptotic sublinearity. The converse fails. For example, $g(x) = x/\log x$ is sublinear for large x but has no asymptotic power growth with a positive finite coefficient, and $x^{1/2}$ and $x^{3/4}$ are both sublinear but encode different exponents. Sublinearity therefore supplies no unique tail-index mapping.

Definition A.2 (Uniform local boundedness)**.** A function $f : \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}_+$ is *locally bounded in the first argument, uniformly in the second*, if $\sup_{0 \leqslant w \leqslant b,\, s \in \mathbb{S}} f(w, s) < \infty$ for every $b > 0$.

For brevity, we call such f *uniformly locally bounded*. Uniformity in s is the only addition to the standard local-boundedness notion of regular variation theory (Bingham et al., 1987, cf. Corollary 1.4.2).

Power transforms and the weak tail mapping

Lemma A.1 (Power transformation of a Pareto-type tail)**.** *Suppose $h > 0$ and $\rho > 0$ are constants, $X \geqslant 0$, and $Y = hX^\rho$. If X has a Pareto-type tail with index $\alpha > 0$, then Y has a Pareto-type tail with index α/ρ. If, in addition, X has a Pareto tail with constant k_X, then Y has a Pareto tail with constant $k_X h^{\alpha/\rho}$. If instead X has a weak Pareto-type tail with index α, then Y has a weak Pareto-type tail with index α/ρ.*

Proof. If $\bar{F}_X(x) = x^{-\alpha} L(x)$ with L slowly varying, then

$$\mathbb{P}\{hX^\rho > x\} = \bar{F}_X\left((x/h)^{1/\rho}\right) = h^{\alpha/\rho} x^{-\alpha/\rho} L\left((x/h)^{1/\rho}\right).$$

The inner map $x \mapsto (x/h)^{1/\rho}$ is regularly varying with index $1/\rho$ and tends to infinity. Its composition with L is therefore slowly varying (Bingham et al., 1987, Proposition 1.5.7(ii)), so Y is Pareto-type with index α/ρ. If $x^\alpha \bar{F}_X(x) \to k_X$, the same calculation gives the stated Pareto-tail constant. For the weak clause, put $u_x := (x/h)^{1/\rho}$. Then $-\log \bar{F}_Y(x)/\log x = (-\log \bar{F}_X(u_x)/\log u_x)(\log u_x/\log x) \to \alpha/\rho$, since $\log u_x/\log x \to 1/\rho$. □

Lemma A.2 (Sandwiching yields a weak Pareto-type tail)**.** *Let X, Y, Z be nonnegative random variables, and suppose Y and Z have Pareto-type tails with the same index $\tau > 0$. If there exists $x_0 > 0$ such that*

$$\mathbb{P}\{Y > x\} \leqslant \mathbb{P}\{X > x\} \leqslant \mathbb{P}\{Z > x\}$$

for all $x \geqslant x_0$, then X has a weak Pareto-type tail with index τ. In particular, the conclusion holds if $Y \leqslant X \leqslant Z$ almost surely. The conclusion also holds if Y and Z have only weak Pareto-type tails with index τ.

Proof. Pareto-type tails satisfy the logarithmic limit, so $-\log \bar{F}_Y(x)/\log x \to \tau$ and likewise for Z. The survival ordering squeezes $-\log \bar{F}_X(x)/\log x$ between them. If Y

and Z have only weak Pareto-type tails, the logarithmic limits hold by definition and the same squeeze applies. □

Theorem A.3 (Uniform asymptotic power growth and weak tail mapping)**.** *Suppose f is uniformly locally bounded (Definition A.2) and has uniform asymptotic power growth with exponent $\rho > 0$ and limit function $H : \mathbb{S} \to (0, \infty)$ in the sense of Definition A.1(iv). If W has a weak Pareto-type tail with index $\alpha_W > 0$, then C has a weak Pareto-type tail with index α_W/ρ. Writing this weak index as α_C gives*

$$\alpha_C = \frac{\alpha_W}{\rho}. \tag{19}$$

Proof. Let $H_{\min} := \inf_{s\in\mathbb{S}} H(s)$ and $H_{\max} := \sup_{s\in\mathbb{S}} H(s)$. Definition A.1(iv) gives $0 < H_{\min} \leqslant H_{\max} < \infty$. Fix $\varepsilon \in (0, H_{\min}/2)$. By uniform convergence, there is $w_0 > 0$ such that, for all $w \geqslant w_0$ and all s,

$$(H(s) - \varepsilon)w^\rho \leqslant f(w, s) \leqslant (H(s) + \varepsilon)w^\rho.$$

Hence, for all $w \geqslant w_0$ and all s,

$$\frac{H_{\min}}{2} w^\rho \leqslant f(w, s) \leqslant 2H_{\max} w^\rho.$$

Local boundedness gives $\bar{f}_{w_0} := \sup_{w\leqslant w_0, s\in\mathbb{S}} f(w, s) < \infty$. Let $Y = (H_{\min}/2)W^\rho$ and $Z = 2H_{\max}W^\rho$. Set $x_0 := \max\{\bar{f}_{w_0}, (H_{\min}/2)w_0^\rho\}$. For every $x > x_0$, the event $\{Y > x\}$ implies $W > w_0$, so the lower policy bound gives $C \geqslant Y$. The event $\{C > x\}$ also implies $W > w_0$ because $C \leqslant \bar{f}_{w_0}$ when $W \leqslant w_0$. The upper policy bound then gives $C \leqslant Z$. Hence

$$\mathbb{P}\{Y > x\} \leqslant \mathbb{P}\{C > x\} \leqslant \mathbb{P}\{Z > x\} \qquad \text{for all } x > x_0.$$

By the weak clause of Lemma A.1, both Y and Z have weak Pareto-type tails with index α_W/ρ, and Lemma A.2 completes the proof. □

The theorem permits arbitrary dependence between W and S, which matters in the economic applications, where stationary wealth and the idiosyncratic state are generally dependent. The price of that generality is a weaker conclusion: C is only weak Pareto-type, though the theorem likewise needs only a weak index for W. The theorem gives a policy-level weak-tail formulation related to the exact index mappings of Gaillard et al. (2023, Propositions 1 and 3) and to the policy limit of Ma and Toda (2025, Theorem 2), whose quantitative section applies that limit to a calibrated wealth distribution.

Pareto tail constants

Independent multiplicative factors are handled by Breiman's lemma (Breiman, 1965; Resnick, 2007). If X has a Pareto-type tail with index α_X, M is a positive random variable independent of X, and $\mathbb{E}[M^{\alpha_X+\varepsilon}] < \infty$ for some $\varepsilon > 0$, then $\bar{F}_{MX}(x) \sim \mathbb{E}[M^{\alpha_X}]\bar{F}_X(x)$. Multiplication by M therefore scales the tail by $\mathbb{E}[M^{\alpha_X}]$ and leaves the Pareto-type index unchanged. The next result applies this product form.

Corollary A.4 (Pareto tail constant). *Suppose the conditions of Theorem A.3 hold. If, in addition, W has a Pareto tail with index α_W and constant k_W and W is independent of $H(S)$, then $C = f(W,S)$ has a Pareto tail with index α_W/ρ and constant*

$$k_C = k_W \, \mathbb{E}\left[H(S)^{\alpha_W/\rho}\right]. \tag{20}$$

Proof. Set $\mathcal{G} := H(S)W^\rho$, and let $H_{\min}$ and $H_{\max}$ be as in the proof of Theorem A.3. For each $s \in \mathbb{S}$, uniform convergence gives $H(s) = \lim_{n\to\infty} f(n,s)/n^\rho$ along integer n. Hence H is measurable as a pointwise limit of the measurable sections $s \mapsto f(n,s)/n^\rho$. Since W is independent of $H(S)$ and W^ρ is a measurable function of W, the variables W^ρ and $H(S)$ are independent. By Lemma A.1, W^ρ has a Pareto tail with index α_W/ρ and constant k_W. Moreover, $H(S)$ is positive and bounded. Breiman's lemma, stated above, therefore gives

$$x^{\alpha_W/\rho}\mathbb{P}\{\mathcal{G} > x\} \to k_{\mathcal{G}} := k_W \mathbb{E}\left[H(S)^{\alpha_W/\rho}\right].$$

Fix $\delta \in (0,1)$. By uniform convergence and $H_{\min} > 0$, there is $w_\delta > 0$ such that

$$\left|\frac{f(w,s)}{w^\rho} - H(s)\right| \leqslant \delta H_{\min} \leqslant \delta H(s) \qquad \text{for all } w \geqslant w_\delta \text{ and all } s,$$

and hence

$$(1-\delta)H(s)w^\rho \leqslant f(w,s) \leqslant (1+\delta)H(s)w^\rho \qquad \text{for all } w \geqslant w_\delta \text{ and all } s.$$

Set $\bar{f}_{w_\delta} := \sup_{w\leqslant w_\delta, s\in\mathbb{S}} f(w,s)$ and $x_\delta := \max\{\bar{f}_{w_\delta}, (1-\delta)H_{\max}w_\delta^\rho\}$. Local boundedness gives $\bar{f}_{w_\delta} < \infty$. For every $x \geqslant x_\delta$, the event $\{(1-\delta)\mathcal{G} > x\}$ implies $H(S)W^\rho > H_{\max}w_\delta^\rho$, and since $H(S) \leqslant H_{\max}$ this forces $W > w_\delta$. The lower policy bound then gives $C \geqslant (1-\delta)\mathcal{G} > x$. The event $\{C > x\}$ also implies $W > w_\delta$ because $C \leqslant \bar{f}_{w_\delta}$ when $W \leqslant w_\delta$. The upper policy bound then gives $C \leqslant (1+\delta)\mathcal{G}$. Hence

$$\mathbb{P}\{(1-\delta)\mathcal{G} > x\} \leqslant \bar{F}_C(x) \leqslant \mathbb{P}\{(1+\delta)\mathcal{G} > x\} \qquad \text{for all } x \geqslant x_\delta.$$

For every constant $\chi > 0$,

$$x^{\alpha_W/\rho}\mathbb{P}\{\chi\mathcal{G} > x\} = \chi^{\alpha_W/\rho}(x/\chi)^{\alpha_W/\rho}\mathbb{P}\{\mathcal{G} > x/\chi\} \to \chi^{\alpha_W/\rho}k_{\mathcal{G}}.$$

Applying this with $\chi = 1 - \delta$ and $\chi = 1 + \delta$ gives

$$(1-\delta)^{\alpha_W/\rho}k_{\mathcal{G}} \leqslant \liminf_{x\to\infty} x^{\alpha_W/\rho}\bar{F}_C(x) \leqslant \limsup_{x\to\infty} x^{\alpha_W/\rho}\bar{F}_C(x) \leqslant (1+\delta)^{\alpha_W/\rho}k_{\mathcal{G}}.$$

Letting δ decrease to zero proves (20). □

Two sufficient conditions cover the applications. If W is independent of S, then W is independent of $H(S)$, because $H(S)$ is a measurable function of S. If the limit function is constant, $H(s) = m > 0$ for all s, independence holds automatically and the constant is $k_C = k_W m^{\alpha_W/\rho}$, with no restriction on the dependence between W and S.

Linear bounds and tail domination

A one-sided upper bound gives complementary conclusions under less structure: beyond a threshold, consumption does not exceed wealth, the budget feasibility $c \leqslant w$.

Condition A.1 (Linear upper bound)**.** There exists $w_1 \geqslant 0$ such that

$$f(w,s) \leqslant w \qquad \text{for all } w \geqslant w_1 \text{ and all } s \in \mathbb{S}.$$

When w denotes resources available before consumption, Condition A.1 is the budget bound. The witness $w_1 = 0$ is the global case, $f(w,s) \leqslant w$ for all $w \geqslant 0$, which forces $f(0,s) = 0$, implies uniform local boundedness, and implies $\mathbb{P}\{C \leqslant W\} = 1$. Models with another timing or resource convention must verify the relevant bound directly.

Definition A.3 (Eventual tail domination)**.** *W eventually tail-dominates C* if there exists $x_0 \geqslant 0$ such that $\bar{F}_C(x) \leqslant \bar{F}_W(x)$ for every $x \geqslant x_0$.

For $x_0 = 0$, C is smaller than W in the usual stochastic order (cf. Shaked and Shanthikumar, 2007, Section 1.A.1, eq. (1.A.1)). The eventual form is what the proofs use.

Lemma A.5 (Two sufficient conditions for tail domination)**.** *(i) If Condition A.1 holds and f is uniformly locally bounded, then W eventually tail-dominates C.*

(ii) If $\mathbb{P}\{C \leqslant W\} = 1$, then W eventually tail-dominates C with $x_0 = 0$.

Proof. (i) Choose $b > w_1$ and set $\bar{f}_b := \sup_{0 \leqslant w \leqslant b,\, s \in \mathbb{S}} f(w,s)$, which is finite by Definition A.2. For $x > \max\{\bar{f}_b, b\}$, the event $\{C > x\}$ implies $W > b$: otherwise $C \leqslant \bar{f}_b < x$. On $\{W > b\}$, Condition A.1 gives $C \leqslant W$, since $b \geqslant w_1$, so $W \geqslant C > x$. Hence $\bar{F}_C(x) \leqslant \bar{F}_W(x)$ for every $x > \max\{\bar{f}_b, b\}$. (ii) If $\mathbb{P}\{C \leqslant W\} = 1$, then $\{C > x\} \subseteq \{W > x\}$ up to a null event for every $x \geqslant 0$, so $\bar{F}_C(x) \leqslant \bar{F}_W(x)$ for every $x \geqslant 0$. □

Corollary A.6 (Two-sided bounds give weak tail-index inheritance)**.** *Suppose W eventually tail-dominates C and Assumption 3.1 holds. If W has a weak Pareto-type tail with index α_W, then C has a weak Pareto-type tail with index α_W.*

Proof. As in the proof of Theorem 3.1, $\bar{F}_W(x/a) \leqslant \bar{F}_C(x)$ for $x \geqslant a w_0$, and domination gives $\bar{F}_C(x) \leqslant \bar{F}_W(x)$ for $x \geqslant x_0$. Both outer logarithmic ratios tend to α_W, and the squeeze gives the limit for C. □

Uniformity details for Proposition 3.3

Proposition 3.3 follows from Corollary A.4 once the policy is shown to have uniform asymptotic power growth with exponent one. The remainder of this appendix verifies that condition under each of the proposition's two policy assumptions.

Exact affine policies. Under Assumption 3.3,

$$\frac{f(w,s)}{w} = h(s) + \frac{g(s)}{w}.$$

Continuity on compact $\mathbb{S}$ makes g bounded and gives constants $0 < h_{\min} \leqslant h(s) \leqslant h_{\max} < \infty$. Hence

$$\sup_{s \in \mathbb{S}} \left| \frac{f(w,s)}{w} - h(s) \right| \leqslant \frac{\sup_{s \in \mathbb{S}} g(s)}{w} \longrightarrow 0.$$

The policy is locally bounded because, for each $b < \infty$,

$$\sup_{0 \leqslant w \leqslant b,\, s \in \mathbb{S}} f(w,s) \leqslant h_{\max} b + \sup_{s \in \mathbb{S}} g(s) < \infty.$$

Thus the conditions of Corollary A.4 hold with $\rho = 1$ and $H = h$.

Concave asymptotically linear policies. Under Assumption 3.4, define $r_w(s) := f(w,s)/w$ for $w > 0$. Concavity of $f(\cdot, s)$ and $f(0,s) \geqslant 0$ imply that $r_w(s)$ is nonincreasing in w for every s. The sequence of continuous functions r_n therefore decreases

pointwise on compact $\mathbb{S}$ to the continuous function K. Dini's theorem (Rudin, 1976, Theorem 7.13) gives

$$\sup_{s\in\mathbb{S}} |r_n(s) - K(s)| \longrightarrow 0.$$

For any $w \geqslant 1$, let $n = \lfloor w \rfloor$. Since $r_n(s) \geqslant r_w(s) \geqslant K(s)$,

$$0 \leqslant \sup_{s\in\mathbb{S}}\{r_w(s) - K(s)\} \leqslant \sup_{s\in\mathbb{S}}\{r_n(s) - K(s)\} \longrightarrow 0.$$

Thus $f(w,s)/w \to K(s)$ uniformly in s. Continuity and positivity of K on compact $\mathbb{S}$ give $0 < K_{\min} \leqslant K(s) \leqslant K_{\max} < \infty$, while continuity of f on $[0,b] \times \mathbb{S}$ gives local boundedness for each $b < \infty$. Corollary A.4 therefore applies with $\rho = 1$ and $H = K$.

B Model coverage of the impossibility result

This appendix surveys heterogeneous-agent models associated with heavy wealth tails and asks whether their optimal consumption policies satisfy the positive linear lower bound in Assumption 3.1. The mechanisms generating these wealth tails are summarized in Table 7. In the permanent-income row, G' denotes next-period gross permanent-income growth and the multiplier applies to wealth normalized by permanent income, and in the persistent-discount row, β denotes the discount factor. The multipliers are high-wealth coefficients in the sense of Section 4: whether a stationary Pareto tail exists depends on the relevant process and hypotheses there, with the exact-affine matrix of equation (13) and the expansion-only matrix of equation (14) kept distinct.

Table 7: Main mechanisms for Pareto-tailed stationary wealth

Mechanism	Economic interpretation	Multiplier	Reference
Stochastic idiosyncratic returns	Portfolio risk, entrepreneurial income, private business returns	$A = R' s_\infty(S)$	Benhabib et al. (2015)
Persistent discount heterogeneity	Patient households save more persistently than impatient households	$A = R s_\infty(S, \beta)$	Toda (2019)
Permanent-income normalization	Dividing by idiosyncratic permanent-income growth makes the effective return stochastic	$A = (R/G') s_\infty(S)$	Carroll (1997); Ma et al. (2020)
Dynastic or bequest scaling	Intergenerational transmission creates multiplicative wealth propagation	model specific	Benhabib et al. (2011)

Tables 8, 9, and 10 summarize the findings. In the tables, the column "Trapped?" records whether the paper's lower-bound logic restricts the consumption tail. When wealth has a Pareto-type index, Theorem 3.1 rules out $\alpha_C > \alpha_W$. For the nonlinear income-fluctuation problem, where Ma et al. (2020) establish a Pareto-order lower bound rather than an exact wealth-tail index, Theorem 4.2 instead rules out an exact consumption-tail index above the Pareto-order boundary κ_W. A ✓ means Assumption 3.1 is established by a formal result in the cited paper or follows by direct specialization from such a result. A ✓* means Assumption 3.1 is supported by an economic argument (the high-wealth homogeneity reasoning stated below) but has not,

to our knowledge, been formally proved in the cited paper. A $\times$ means Assumption 3.1 fails by a formal result in the cited paper. A reader can therefore distinguish entries where the impossibility or its failure is a theorem from entries where the status is a well-motivated conjecture or open.

Under an exact Pareto wealth tail, independence of stationary wealth W from the exogenous state S, and the affine or concave asymptotically linear policy conditions of Proposition 3.3, that proposition sharpens the impossibility to the exact equality $\alpha_C = \alpha_W$. Appendix C records when the independence condition holds and the weaker tail-factorization alternative.

The high-wealth homogeneity argument. Several table entries marked $\checkmark^*$ rely on the following reasoning, stated once to avoid repetition. Under CRRA utility, the budget constraint (7) becomes approximately homogeneous of degree one in wealth w as $w \to \infty$, because nonfinancial income Y has finite mean (Assumption 4.1(iv)) and is therefore negligible relative to w. The value function inherits this homogeneity, behaving like $w^{1-\gamma}/(1-\gamma)$ times a state-dependent factor at large w, and optimal consumption is approximately $f(w,s) \approx m_\infty(s)\, w$ with $m_\infty(s) > 0$. This argument is formalized for the discrete-time income-fluctuation problem by Ma et al. (2020, Proposition 2.5). Under the conditions of Corollary 4.4, shifted HARA utility retains the linear lower bound, although existence of the asymptotic consumption slope is not established there. For models not directly covered by these results, a formal proof would require verifying the relevant concavity-preservation condition (Ma et al., 2020, condition (17)) or its continuous-time analogue. The $\checkmark^*$ entries below record that this step remains conjectural in the cited settings.

Table 8: Core frameworks: does the impossibility apply?

Model class	Heavy wealth tail?	Trapped?	Why Assumption 3.1 holds
Risk-free saving, constant β, under the conditions of Stachurski and Toda (Bewley, 1976; Huggett, 1993; Aiyagari, 1994)	inherits the income tail (Thms. 3, 8)	✓	Prop. 5 of Stachurski and Toda (2019), as corrected in their 2020 corrigendum, gives the positive linear consumption lower bound
General-equilibrium Bewley economy with i.i.d. stochastic returns, CRRA (Benhabib et al., 2015)	yes (their Thm. 4)	✓	f asymptotically linear with $m_\infty \in (0, 1)$ (Benhabib et al., 2015, Prop. 5)
Stochastic returns, Markov (Ma et al., 2020)	lower bound (their Thm. 3.3)	✓	$f(w, s) \geqslant (1 - \bar{x})w$ (Ma et al., 2020, Prop. 2.6) and is asymptotically linear under CRRA (Ma et al., 2020, Prop. 2.5)
Overlapping generations + stochastic returns (Benhabib et al., 2011)	yes (their Thms. 1–2)	✓	f affine in wealth: $f(w, s) = m(s)(w + p(s))$ (Benhabib et al., 2011, Appendix A). Proposition 4.6 transfers the population power bound to consumption
Random discount factors (Toda, 2019)	yes (their Sec. 3)	✓	$f(w, s) = a_s^{-1/\gamma} w$ (Toda, 2019, Prop. 2.1)

Table 9: Extensions: does the impossibility apply?

Model class	Heavy wealth tail?	Trapped?	Why Assumption 3.1 holds
Shifted HARA utility (Ma and Toda, 2022)	conditional on a separate wealth-tail assumption	✓	Corollary 4.4 delivers the linear lower bound under the Ma and Toda (2022) foundation and the stability condition, without policy concavity
Perpetual youth + returns (Benhabib et al., 2016)	yes (their Prop. 2)	✓	$C(s,t) = A^{-1/\gamma} W(s,t)$ (Benhabib et al., 2016, Prop. 1)
Persistent heterogeneous returns, continuous time (Cao and Luo, 2017)	yes	✓*	CRRA, with the high-wealth homogeneity argument
Stochastic returns + random β (Hubmer et al., 2021)	yes	✓*	Saving rules documented as asymptotically linear (numerical, not proved formally)
Scale-dependent returns extension (Gaillard et al., 2023)	yes	✓*	Their Prop. 1, for homothetic preferences without a wealth motive and under its parameter restriction, gives $c \sim b_n w^{1+\eta}$ with $b_n > 0$: for $\eta > 0$ the policy is superlinear at high wealth, so the linear lower bound holds state by state, with a single threshold uniform across labor-productivity states the remaining step. Their mapping $\alpha_C = \alpha_W/(1+\eta) \leqslant \alpha_W$ is consistent with Theorem 3.1, and the scale-independent case ($\eta = 0$) gives equality (Appendix C)

Table 10: Models that may escape the impossibility.

Model class	Heavy wealth tail?	Trapped?	Why Assumption 3.1 may fail
Preference for wealth ($\delta < \gamma$) (Ma and Toda, 2025)	in their calibrated stochastic-return model	×	$m_\infty = 0$, so Assumption 3.1 fails. Their Theorem 2 gives the policy exponent δ/γ (Ma and Toda, 2025, Thm. 2). Given a weak Pareto-type wealth index and the conditions of Theorem A.3, the weak consumption index is $\alpha_W \gamma/\delta$. See Hu (2026, Chapter 5)
Zero asymptotic MPC under high γ (Ma and Toda, 2021)	their calibrated return-risk example	**possibly not**	In their calibrated return-risk example, sufficiently high risk aversion yields a zero asymptotic MPC. This result alone leaves the consumption tail index open.
Multi-asset / portfolio (Kaplan and Violante, 2014; Gomez, 2025)	open	**open**	Hu (2026, Chapter 5) proves an escape for trade-size liquidation costs whose induced effective consumption cost is regularly varying with index above one under its high-wealth reduction condition. Holdings-scaled convexity alone does not deliver that result, and the generic multi-asset case remains unresolved

C Related results, benchmarks, and open problems

Relationship to Stachurski–Toda and Gaillard et al.

Two prior results are most closely related to the impossibility theorem.

Stachurski–Toda. Stachurski and Toda (2019) prove that in Bewley–Huggett–Aiyagari models with infinitely-lived agents, risk-free saving (R constant), and constant discount factors (β constant), the wealth distribution inherits the tail behavior of income shocks (their Theorems 3 and 8). Their impossibility concerns wealth versus income, whereas ours concerns wealth versus consumption. A direct connection exists. Under its strengthened assumptions, Proposition 5′ of Stachurski and Toda (2020) establishes that optimal consumption satisfies $c(a, z) \geqslant ma$ for a constant $m > 0$. In our notation, this is $f(w, s) \geqslant mw$, exactly Assumption 3.1 of Theorem 3.1 with lower-bound constant m and $w_0 = 0$. Thus, under the corrigendum's assumptions, the Stachurski and Toda (2019) setting satisfies the policy condition used here. The related result of Benhabib et al. (2017), which shows that a class of OLG models cannot jointly match earnings and wealth tails, also operates through the linear structure of optimal consumption (Benhabib et al., 2011, Appendix A).

Gaillard et al. The continuous-time model of Gaillard et al. (2023) combines CRRA utility, stochastic capital income, and perpetual-youth demographics. In their model, consumption and wealth share the same Pareto tail index under homothetic preferences and scale-independent returns (Gaillard et al., 2023, Proposition 1). Their derivation relies on two features of their scale-independent benchmark ($\eta = 0$), for homothetic preferences without a wealth motive and under the parameter restriction of their Proposition 1: (i) the consumption policy is asymptotically linear in wealth (a consequence of CRRA utility and the homogeneous HJB equation in their perpetual-youth setting), and (ii) a tail-factorization property of the joint distribution of wealth and the exogenous state. Property (i) delivers Assumption 3.1 along each state, with uniformity across states the remaining step, exactly as the qualified survey row records (Appendix B). Theorem 3.1 requires neither feature: only Assumption 3.1 and a Pareto-type tail for wealth. The open problems below state the factorization condition and the open problem of extending Proposition 3.3. Appendix B records, model by model, whether the positive linear lower bound is established formally or supported by the high-wealth homogeneity argument across the heterogeneous-agent literature.

Exact affine and multiplicative benchmarks

For an exact affine recursion $W_{t+1} = A_{t+1}W_t + B_{t+1}$ with Markov-dependent coefficients, Markov extensions of Kesten–Goldie theory establish power-law tails under additional moment, irreducibility, and nonarithmetic conditions (Roitershtein, 2007). In the finite-state benchmark used here, the multiplier moments are collected in K_α (13), with entries $Q_{ij}\mathbb{E}[(s_\infty(s_i)R(s_j,\zeta))^\alpha]$. Its spectral crossing is the tail index used in the exact-affine calibration.

Beare and Toda (2022) study a different exact process: Markov multiplicative growth with state-dependent random resets. Their Pareto tail index also solves a spectral-radius equation, but that equation includes the reset or survival probabilities as well as the multiplier moments. The nonlinear income-fluctuation recursion is different again. Theorem 3.3 of Ma et al. (2020) uses the expansion-only matrix K_κ^+ in (14) to establish the Pareto-order lower bound in (15). These spectral conditions are not interchangeable: K_α gives the exact-affine tail index, K_κ^+ only the Pareto-order lower bound for the nonlinear model, and the two need not coincide. An exact-tail result for the nonlinear recursion would require additional arguments.

Open problems

General utility. Proposition 4.1 delivers the linear lower bound $f(w,s) \geqslant (1-\bar{x})w$ for any u satisfying Assumption 4.1, without requiring CRRA or HARA. Corollary 4.4 delivers the same bound for shifted HARA utility, whose finite marginal utility at zero places it outside Assumption 4.1, and its argument uses no policy concavity. The Pareto trap (Theorem 4.2) additionally requires the limit $m_\infty(s) \in (0,1)$, which is established under CRRA (Corollary 4.3). Remark 4.4 states what the shifted-HARA extension does and does not establish. Verifying the concavity-preservation condition (Ma et al., 2020, condition (17)) for the shifted-HARA class is the natural attack.

Tail factorization. Proposition 3.3 requires independence of stationary wealth W and the exogenous state S. In the i.i.d. benchmark of Benhabib et al. (2015), there is no nondegenerate exogenous policy state, so the condition is automatic. In OLG models where each generation's initial state is drawn independently of inherited wealth (Benhabib et al., 2011), it holds at birth, while at later ages current wealth and the within-life state are generally dependent in the pooled population, so the condition must be verified for the population actually compared. I.i.d. innovations alone do not guarantee this independence under every timing convention, and it need not hold

in Markov settings (Ma et al., 2020). Gaillard et al. (2023) use a weaker condition: multiplicative separability of the joint density of (W, S) in the tail (their footnote 27), under which the conditional distribution of S given $\{W > w\}$ converges as $w \to \infty$. An extension of Proposition 3.3 to such tail factorization could cover their asymptotically linear consumption case if the other policy and tail assumptions also hold.

Joint tails. Theorem 3.1 and Proposition 3.3 concern marginal tail indices. The dependence structure between wealth and consumption in the tail is a separate object, characterized by quantities such as the upper-tail dependence coefficient and the stable tail dependence function. Whether the impossibility logic extends to these joint-tail objects is open.

Policy wedges and aggregate risk. Whether progressive taxation or aggregate shocks can violate the positive linear lower bound remains open. The benchmark includes neither feature. Subsection 3.6 states that condition precisely.

Endogenous labor supply and portfolio choice. Endogenous labor supply and multi-asset portfolio choice remain open beyond the mechanisms studied in Hu (2026, Chapter 5). In multi-asset models (Kaplan and Violante, 2014; Gomez, 2025), consumption depends on the composition of wealth, not just its level. Under the high-wealth reduction in Hu (2026, Chapter 5), a trade-size cost with effective-cost index above one yields sublinear consumption. In the one-asset benchmark, an asymptotically linear effective cost preserves the weak tail index under the chapter's stationarity and positive limiting expenditure-share assumptions. For the holdings-scaled cost considered there, cost is proportional to the position at a fixed trade share. This scaling alone leaves the optimal consumption policy open. Whether other portfolio structures generate sublinearity remains open. Similarly, if labor supply responds to wealth, the effective return on saving may change at the top, potentially altering the asymptotic form of the consumption rule.

Continuous-time formal proofs. The high-wealth homogeneity argument suggests that consumption is asymptotically linear in continuous-time HA models with stochastic returns (Gabaix, 2016; Cao and Luo, 2017). Establishing this claim under explicit conditions, through a proof analogous to Ma et al. (2020, Proposition 2.5), would place these models on the same formal footing as the discrete-time cases.

D Remarks, data, methodology and robustness checks

D.1 Remarks on Pareto tail definition

Remark D.1. In practice, for large values of x, we plot $\ln \bar{F}_X(x)$ against $\ln x$ (a *log-log* or *double-log plot*) to assess visually whether X exhibits a Pareto tail. By Definition 2.1, if F_X has a Pareto tail with index $\alpha > 0$, then for some constant $k > 0$ and sufficiently large x, $\bar{F}_X(x) \simeq kx^{-\alpha}$. Taking logarithms gives $\ln \bar{F}_X(x) \simeq \ln k - \alpha \ln x$, so the CCDF appears as a straight line in the double-log plot. Figure 6 illustrates this for two Pareto tails with indices 1.5 and 2.5.

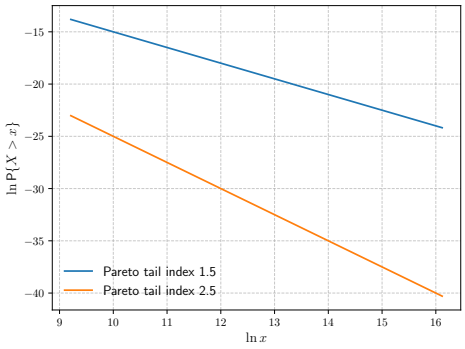

Figure 6: Log-log CCDFs: Pareto tails with indices 1.5 and 2.5

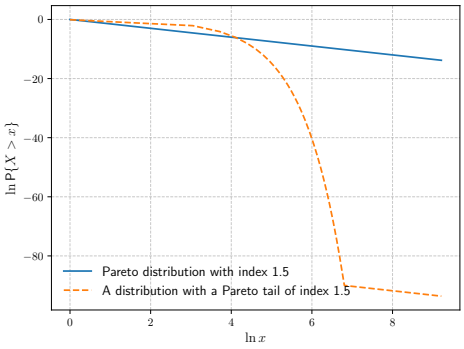

Figure 7: A distribution with a Pareto tail (index 1.5) that is not globally Pareto

Remark D.2. The Pareto tail index measures the heaviness of a distribution's tail. As illustrated in Figure 6, a lower index indicates a heavier tail: extremely large values are more prevalent, a greater share of the distribution is concentrated among the top percentile, and inequality at the top is higher.

Remark D.3. A Pareto tail does not imply a Pareto distribution. Figure 7 shows a distribution F_X that follows an exponential distribution with rate $\lambda = 0.1$ for $0 < x < x_m = 900$ and a Pareto tail with index $\alpha = 1.5$ for $x \geqslant x_m$: specifically, $\bar{F}_X(x) = kx^{-\alpha}$ with $k := \exp(-\lambda x_m) x_m^{\alpha}$ for $x \geqslant x_m$, alongside a pure Pareto distribution with the same index.

D.2 Eligibility and multiple comparisons

An index pair enters asymptotic comparisons only if it passes four checks fixed before estimation.

(i) *Construction.* The consumption and wealth variables pass the audited construction rules on the matched household sample, with documented treatment of missing values.

(ii) *Fit.* The Pareto model is an adequate description above the fixed threshold. Adequacy is assessed by a bootstrap goodness-of-fit test at the released threshold, and a failing margin disqualifies the pair.

(iii) *Threshold stability.* The fitted index is stable as the threshold moves across the prespecified quantile window from the 85th to the 95th percentile.

(iv) *Inference.* The design-based uncertainty is well defined, with at least 100 raw and 50 effective tail observations behind each fit.

Ineligible pairs remain in the family with a p-value of one. The seven one-sided tests are combined by Holm's step-down rule (Holm, 1979). Order the raw p-values from smallest to largest, compare the smallest to $0.05/7$, the next to $0.05/6$, and so on, and stop at the first failure. The rule holds the familywise error rate at 5 percent whatever the dependence between the tests. The exact-tail benchmark is stricter than eligibility. It additionally requires an eligible Pareto fit on both margins with both indices above one, and it is the only status that licenses calculations assuming an exact Pareto distribution.

D.3 Data

The data used in this study include waves 2, 6–10, 14, and 18 of the Household, Income and Labour Dynamics in Australia (HILDA) survey and the 2007, 2013, and 2018 waves of the Chinese Household Income Project Survey (CHIPS). HILDA is an annual survey conducted in Australia. CHIPS collects similar information in China. These datasets were selected because they are among the few that provide both wealth and consumption information in their respective countries.

We define wealth and consumption at the household level as follows. Wealth is net worth: the difference between total assets and total liabilities. Consumption in HILDA is the sum of all nondurable goods expenditures available in each wave. For CHIPS, consumption is defined using the consumption expenses variable (*n4202*), as specified in the survey.

Using these definitions, we construct household-level distributions of wealth and consumption for each wave in HILDA and CHIPS through aggregation and deflation. We also construct individual-level distributions for HILDA by applying adult equivalization. Details of the aggregation, deflation, and equivalization processes are in Appendix D.4.

Wave number	6	7	8	9	10	pooled
Sample size	7139	7063	7066	7234	7317	35819
Min	0.00	7.67	0.00	0.00	0.00	0.00
Max	2432.5	1950.8	5080.4	2928.1	2440.8	5080.4
Mean	143.8	147.5	151.3	144.0	148.1	147.0
Median	122.2	125.1	127.5	124.0	125.8	124.9
Standard deviation	108.3	104.2	125.0	98.2	110.9	109.7
Skewness	6.04	4.75	12.24	6.24	6.46	7.82
Excess kurtosis	74.16	45.72	376.58	112.85	88.04	180.57
p90	236.6	246.4	250.2	239.7	245.5	243.7
p95	296.1	303.6	319.3	302.0	304.4	305.0
p99	522.2	515.9	558.3	466.1	531.1	518.5
Top 10% share	0.252	0.247	0.256	0.241	0.250	0.249
Top 1% share	0.0584	0.0546	0.0612	0.0491	0.0572	0.0561
Top 0.1% share	0.0130	0.0109	0.0147	0.0109	0.0136	0.0119

Table 11: Descriptive statistics of HILDA nondurable consumption (deflated and adult-equivalized). Kurtosis reported is excess kurtosis (i.e., minus 3).

Wave number	2	6	10	14	18	pooled
Sample size	6889	6784	6940	8978	9116	38707
Min	0.0039	0.0053	0.0074	0.0038	0.0039	0.0038
Max	47516	61566	60008	57259	59411	61566
Mean	2364.0	3223.3	3172.5	3189.6	3619.6	3146.8
Median	1288.4	1710.9	1713.0	1724.3	1970.3	1673.2
Standard deviation	3767.3	5702.9	5232.1	5094.4	5475.8	5135.8
Skewness	4.93	5.31	5.09	4.80	4.17	4.91
Excess kurtosis	35.90	37.73	37.43	34.22	25.81	35.15
p90	5283.3	7018.3	7272.5	7491.2	8618.5	7218.8
p95	8005.9	10744.8	10656.8	11065.2	12523.8	10872.8
p99	19297.5	39234.6	29382.0	25335.2	29089.8	27789.3
Top 10% share	0.451	0.480	0.459	0.454	0.444	0.46
Top 1% share	0.121	0.138	0.127	0.121	0.110	0.12
Top 0.1% share	0.01937	0.01971	0.01908	0.01800	0.01801	0.019

Table 12: Descriptive statistics of HILDA net worth (deflated and adult-equivalized). Kurtosis reported is excess kurtosis (i.e., minus 3).

The descriptive evidence already points toward thinner consumption tails and heavier wealth tails.

Wave number	2007	2013	2018	pooled
Sample size	15746	18948	10754	45448
Min	0.0010	0.0064	0.01	0.0010
Max	7560	6900	3240	7560
Mean	90.54	156.49	195.22	139.67
Median	67.38	119.35	133.49	95.41
Standard deviation	96.81	141.32	168.72	138.56
Skewness	14.24	7.08	3.45	8.73
Excess kurtosis	577.62	126.45	22.26	178.60
p90	179.88	302.61	371.43	270.03
p95	229.30	394.10	506.53	367.62
p99	406.97	754.84	908.37	706.47
Top 10% share	0.30	0.29	0.29	0.30
Top 1% share	0.068	0.069	0.070	0.073
Top 0.1% share	0.014	0.014	0.013	0.014

Table 13: Descriptive statistics of CHIPS consumption (deflated)

Wave number	Pooled (2013 & 2018)		
Wealth	Rural	Urban	Total
Sample size	10406	13309	23715
Min	0.74	0.37	0.37
Max	342468	217068	342468
Mean	928	1736	1381
Median	431.64	813.16	599.45
Standard deviation	3989	4564	4340
Skewness	63.26	20.52	34.98
Excess kurtosis	5211.57	690.8	2101.4
p90	1921.2	3648.8	2904.2
p95	2893.1	5555.6	4513.2
p99	6886.68	15266.16	11974.01
Top 10% share	0.503	0.50	0.52
Top 1% share	0.198	0.182	0.196
Top 0.1% share	0.087	0.061	0.069

Table 14: Descriptive statistics of CHIPS pooled wealth ($W > 0$)

The Chinese results reinforce the Australian pattern.

D.4 Processing the data: aggregation, deflation and equivalization

D.4.1 Aggregation

We construct net worth and consumption for each household by aggregating the corresponding micro-data.

In the HILDA survey, only waves 2, 6, 10, 14, and 18 (denoted by the letters *b*, *f*, *j*, *n*, and *r* respectively) contain wealth-related variables. The variables $\chi hwnwip$ and $\chi hwnwin$ represent the positive and negative parts of household net worth, respectively, where χ is replaced with the letter corresponding to the wave. A household's signed net worth is their difference, $\chi hwnwip$ minus $\chi hwnwin$.

Only HILDA waves from 2006 to 2010 include comprehensive lists of both nondurable and durable consumption variables. Table 15 summarizes all variables representing nondurable and durable consumption. For each household, nondurable consumption is the sum of all nondurable variables, and total consumption is the sum of all nondurable and durable variables listed in Table 15.

Nondurables	Nondurables	Durables
$\chi hxygrci$	$\chi hxytlii$	$\chi hxyncri$
$\chi hxyalci$	$\chi hxyholi$	$\chi hxyucri$
$\chi hxycigi$	$\chi hxyphii$	$\chi hxycmpi$
$\chi hxypbti$	$\chi hxyoii$	$\chi hxytvi$
$\chi hxymli$	$\chi hxyhlpi$	$\chi hxywgi$ [2]
$\chi hxymvfi$	$\chi hxyphmi$	$\chi hxyfrni$
$\chi hxymcfi$	$\chi hxyutli$	
$\chi hxywcfi$	$\chi hxyhmri$	
$\chi hxyccfi$	$\chi hxymvri$	
$\chi hxyedci$		

Table 15: Consumption variables in HILDA

In the CHIPS survey, only waves from 2007, 2013, and 2018 provide comprehensive consumption information. For each household, consumption is defined using the consumption expenses variable (*n4202*), and wealth is the difference between total assets and total liabilities. Table 16 lists all asset and debt variables for both urban and rural households in CHIPS.

[2] χ should be replaced by the letter representing the corresponding waves 6–10.

For an urban household, net worth is the difference between total assets (the sum of all variables in column 1 of Table 16) and total liabilities (column 3). For a rural household, net worth is defined analogously using column 2 for assets and column 3 for liabilities. Pooling urban and rural households for each wave yields the household-level wealth distribution.

Urban assets	Rural assets	Urban/Rural debt
f03	*f03*	*f05*
f04	*f04*	
f08	*f08*	
f07_1	*f07_1*	
f07_2	*f06_1*	
f06_1	*f06_2*	
f06_2	*f06_3*	
f06_3	*f06_4*	
f06_4	*f06_5*	
f06_5	*f06_6*	
f06_6		

Table 16: Assets and debt variables for urban and rural households in CHIPS

D.4.2 Deflation

We deflate the net worth and consumption values for each household in each wave of HILDA and CHIPS to make values across waves comparable, enabling construction of pooled distributions. We use the OECD "Consumer Price Index: Total" as the price index for both HILDA (Australia, variable *AUSCPIALLAINMEI* in FRED[3]) and CHIPS (China, variable *CHNCPIALLMINMEI* in FRED[4]). We normalize the price level to 100 in the year 2000 for Australia and 2007 for China, then divide each wave's values by the corresponding normalized price. This yields deflated household-level distributions of net worth and consumption for each wave.

D.4.3 Equivalization

To obtain individual-level distributions, we divide household wealth or consumption by the number of adult equivalents in the household. Following Kocherlakota and Pistaferri (2009), the number of adult equivalents is $(A+\theta_1 K)^{\theta_2}$, where A is the number

[3]See https://fred.stlouisfed.org/series/AUSCPIALLAINMEI
[4]See https://fred.stlouisfed.org/series/CHNCPIALLMINMEI

of adults, K is the number of children, and θ_1, θ_2 are parameters. Consistent with the existing literature, we set $\theta_1 = 0.7$ and $\theta_2 = 0.65$.

We apply this step only to HILDA data, where the household adult count (variable *χhhadult*) and total household size (*χhhpers*) are directly available. The number of children is imputed as *χhhpers* − *χhhadult*. Individual net worth or consumption is then household value divided by the number of adult equivalents.

D.5 Estimation approach and hypothesis testing

This section provides a brief overview of the Maximum Likelihood Estimation (MLE) method and the hypothesis testing approach used in Clauset et al. (2009).

Let X be a random variable with distribution F_X and complementary CDF $\bar{F}_X(x) := 1 - F_X(x)$. Assume that F_X has a Pareto tail with index $\alpha > 0$ on $[x_{\min}, \infty)$ for some sufficiently large $x_{\min}$ (the *lower bound* of the Pareto tail). Specifically, for $x \geqslant x_{\min}$, $\bar{F}_X(x) \simeq kx^{-\alpha}$ for some constant $k > 0$.

Let x_i, $i = 1, \ldots, n$, be independently and identically distributed observations from X on $[x_{\min}, \infty)$.

D.5.1 Estimation of the tail index

Given $x_{\min}$, the Maximum Likelihood estimator for α is

$$\hat{\alpha} = n \left[\sum_{i=1}^{n} \ln \frac{x_i}{x_{\min}} \right]^{-1}.$$

The standard error of $\hat{\alpha}$ is

$$\sigma \approx \frac{\hat{\alpha}}{\sqrt{n}}.$$

D.5.2 Estimation of the lower bound

Let $S(x)$ be the empirical CDF of the x_i, and let $P(x)$ be the CDF of a Pareto distribution with tail index α above $x_{\min}$. The Kolmogorov–Smirnov (KS) distance between S and P on $[x_{\min}, \infty)$ is

$$D(x_{\min}, S, P) = \max_{x \geqslant x_{\min}} |S(x) - P(x)|. \tag{21}$$

The estimator $\hat{x}_{\min}$ minimizes the KS distance:

$$\hat{x}_{\min} = \arg \min_{x_{\min} \in \mathbb{R}_+} D(x_{\min}, S, P).$$

D.5.3 Hypothesis testing

The hypothesis testing procedure follows Clauset et al. (2009): bootstrap synthetic datasets are generated from the fitted Pareto model, each is re-fitted, and the p-value is the fraction of synthetic KS distances exceeding the empirical one. See Clauset et al. (2009) for details.

D.6 Robustness checks

This appendix reports earlier sensitivity calculations for HILDA and CHIP. It compares goodness-of-fit measures, consumption definitions, and tail estimators, and records a separate HILDA weighting comparison. Section 2 gives the wave-specific estimates and survey inference used in the paper. All wealth-side comparisons inherit the General Release top-coding and CHIPS reconstruction limitations stated in Section 2.[5]

HILDA robustness

The robustness results for HILDA are in Tables 17, 18, and 19.

Waves Tail index estimates	6	7	8	9	10	pooled
Nondurable consumption (MLE)	2.90	3.08	2.92	3.05	2.94	2.99
Standard error	0.073	0.095	0.079	0.08	0.079	0.038
Lower bound	1.25	1.46	1.33	1.31	1.33	1.37
KS based p-values	0.274	0.256	0.192	0.022	0.096	0.014
RWKS based p-values	0.128	0.018	0.06	0.252	0.304	0.044
Total consumption (MLE)	2.76	3.03	2.85	3.23	2.97	2.96
Standard error	0.095	0.105	0.113	0.142	0.111	0.053
Lower bound	1.64	1.69	1.82	1.99	1.77	1.84
KS based p-values	0.124	0.546	0.482	0.4	0.092	0.942
RWKS based p-values	0.036	0.196	0.0	0.128	0.008	0.002

Table 17: Robustness check for HILDA consumption

[5] Bootstrap p-values vary across replications due to simulation randomness. Minor differences between the KS-based p-values reported in this appendix and those in the superseded draft wave tables reflect different bootstrap draws and do not materially change the tabulated comparisons.

Wave number Tail index estimates	2	6	10	14	18	pooled
Net worth (MLE)	1.72	1.58	1.80	1.95	1.96	1.80
Standard error	0.050	0.050	0.060	0.066	0.063	0.027
Lower bounds	1.62	1.65	1.99	2.42	2.31	2.12
KS based p-values	0.028	0.008	0.13	0.01	0.0	0.0
RWKS based p-values	0.01	0.006	0.002	0.002	0.0	0.0

Table 18: Robustness check of HILDA wealth

Wave number Tail index estimates	2	6	7	8	9	10	14	18	pooled
Nondurables (OLS)		2.83	2.97	2.78	3.17	2.88			2.94
Lower bound		1.25	1.46	1.33	1.31	1.33			1.37
$g(n)$		0.992	0.992	0.990	0.988	0.991			0.998
Net worth (OLS)	1.74	1.47				1.69	1.80	1.88	1.89
Lower bound	1.62	1.65				1.99	2.42	2.31	2.12
$g(n)$	0.994	0.993				0.992	0.992	0.993	0.998

Table 19: Robustness checks for HILDA consumption and wealth using OLS

In the superseded draft results, consumption is measured by nondurable goods and p-values are computed using the Kolmogorov–Smirnov distance D defined in (21). The robustness checks also use p-values based on the reweighted Kolmogorov–Smirnov (RWKS) distance

$$D^*(x_{\min}, S, P) = \max_{x \geqslant x_{\min}} \frac{|S(x) - P(x)|}{\sqrt{P(x)(1 - P(x))}},$$

and repeat the exercise with total consumption, defined as nondurables plus durables in Appendix D.4.1.

Table 17 shows that the tail index estimates for total consumption are similar to those for nondurable consumption. The RWKS diagnostics are mixed: at the 5% level they reject the fitted Pareto specification for nondurable consumption in wave 7 and in the pooled sample, but not in the other reported waves. The HILDA wealth diagnostics in Table 18 reject more often, reinforcing the need to treat the Pareto approximation as local to the selected thresholds rather than as an exact distributional model.

Table 19 presents OLS-based tail index estimates using the order-statistic regression of Santos and Cajueiro (2024), with lower bounds from the superseded draft wave table. The bias measure $g(n) \in [0, 1]$ of Santos and Cajueiro (2024) quantifies

[6] KS means Kolmogorov–Smirnov, and RWKS means reweighted Kolmogorov–Smirnov.

finite-sample bias, with values near 1 indicating small bias. The reported values range from 0.988 to 0.998. Although OLS estimates are generally slightly smaller than MLE, the current estimates preserve $\hat{\alpha}_C > \hat{\alpha}_W$.

CHIPS robustness

Tables 20 and 21 present the robustness checks for CHIPS.

Wave number Tail index estimates	2013	2018	pooled
Wealth	1.80	1.63	1.69
Standard error	0.044	0.0440	0.041
Lower bound	2.00	2.47	2.62
KS based p-values	0.028	0.682	0.112
RWKS based p-values	0.0	0.576	0.25
Wealth (urban)	1.87	1.58	1.66
Standard error	0.08	0.048	0.045
Lower bound	1.78	1.90	2.09
KS based p-values	0.014	0.666	0.282
RWKS based p-values	0.0	0.566	0.12
Wealth (rural)	1.58	1.89	1.81
Standard error	0.1	0.08	0.067
Lower bound	1.97	2.67	2.61
KS based p-values	0.448	0.86	0.426
RWKS based p-values	0.0	0.0	0.0

Table 20: Robustness check of CHIPS wealth

Wave number Tail index estimates	2007	2013	2018	pooled
Consumption (OLS)	2.47	2.96	3.54	2.99
Lower bound	1.59	2.11	2.96	2.13
$g(n)$	0.993	0.994	0.990	0.998
Wealth (OLS)		1.85	1.62	1.67
Lower bound		2.33	2.47	2.80
$g(n)$		0.9892	0.999	0.99

Table 21: Robustness checks for CHIPS using OLS

Table 20 presents wealth tail index estimates for urban and rural households in

CHIPS separately. The estimates are similar to those for all households. The RWKS-based p-values are consistent with the corresponding KS-based p-values.

Table 21 presents OLS-based tail index estimates for wealth and consumption in CHIPS, using the lower bounds from the superseded draft wave table. Within the current reconstruction, all wealth tail estimates are smaller than the corresponding consumption estimates, so the ranking $\hat{\alpha}_C > \hat{\alpha}_W$ is preserved under this alternative estimation method. The $g(n)$ values are close to one.

HILDA weighted MLE robustness

The earlier HILDA General Release comparison pools waves 6–10 for consumption and waves 2, 6, 10, 14, and 18 for wealth. Each Pareto fit selects its lower threshold by minimizing the Kolmogorov–Smirnov distance. Both wealth fits use the reported amounts without a censoring term. Section 2 estimates the two margins in wave 22 above their respective 90th percentiles and treats top-coded wealth as right-censored.

Table 22 compares weighted and unweighted fits on the same pooled observations within each margin. Both sets of point estimates give a higher consumption index than wealth index.

The CHIP checks above include pooled estimates. Section 2 uses equal household weights and reports CHIP 2013 and 2018 separately.

Table 22: HILDA pooled Pareto tail indices with and without cross-sectional household weights. Parentheses give marginal standard errors. The scaled difference divides the weighted estimate minus the unweighted estimate by the square root of the sum of their squared standard errors. This descriptive scale omits the covariance between fits on the same sample.

Country	Variable	$\hat{\alpha}$ (s.e.)	$\hat{\alpha}_{\text{wtd}}$ (s.e.)	Scaled difference	$\hat{\alpha}_C > \hat{\alpha}_W$?
AU	Consumption	3.001 (0.041)	3.060 (0.049)	0.93	Yes
AU	Wealth	1.764 (0.027)	1.784 (0.033)	0.45	

E Distribution-iteration validation

The exact-affine exercise in Subsection 5.4 replaces the nonlinear saving rule $w - f(w)$ with $s_{\text{term}} w$ at every wealth level, where s_{term} is the saving rate at the last policy-grid point. This appendix describes the independent stationary-distribution calculation used to assess that approximation for the Australian calibration.

Transition operators. For each upper wealth bound in $\{10^5, 10^6, 10^7\}$, wealth is discretized on a logarithmic grid from 0.01 at 1,000 points per base-10 decade. The resulting grids contain 7,000, 8,000, and 9,000 points. The full process uses

$$W_{t+1} = R_{t+1}\{W_t - f(W_t)\} + Y_{t+1},$$

where f is the validated endogenous-grid policy. The comparison process uses

$$W_{t+1} = R_{t+1} s_{\text{term}} W_t + Y_{t+1}.$$

The Australian terminal saving rate is 0.9752056, compared with the analytical limit 0.9752066. It is held fixed across the three distribution ceilings. Both processes use the solved nonlinear consumption policy to report consumption moments. Both operators use 25-point Gauss–Hermite quadrature for each shock and linear allocation between adjacent log-wealth grid points. Probability that would leave the grid is assigned to the nearest boundary and recorded before column normalization.

Stationary solution and checks. The leading stationary eigenvector is computed by ARPACK (Lehoucq et al., 1998) and then refined by power iteration, repeated application of the transition operator that sharpens the leading eigenvector. On the primary 10^6 grid, both operators pass every prespecified numerical check. These cover column sums, stationarity, boundary flow, nonnegative probability, and total probability. Repeating the calculation at the two alternative upper bounds leaves the reported body moments effectively unchanged. Sensitivity appears only at probabilities close to the numerical boundary, which is why Figure 4 reports both the primary profile and the range across grid ceilings.

Interpretation. The full nonlinear stationary distribution is much more compact than the exact-affine comparison at ordinary tail fractions. Its local tail index varies with the tail fraction. At the lowest survival probability shown, 10^{-8}, the primary-grid nonlinear exponent is still 3.81 against the calibrated Kesten target of 2.119. The calculation therefore measures the finite-range cost of imposing the limiting policy globally. It neither disproves the Kesten limit nor establishes convergence beyond the observable numerical grid.